Bayesian AR model with Laplace innovations. Results and further proposals.

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Modelling time series

$$Y = (Y_1, Y_2, \dots, Y_T) \tag{1}$$

Assumptions:

- All quantities have density.
- Relation between data is described by model $f(Y_t|\Theta, \mathcal{F}_{t-1})$ and time-independent parameters Θ

$$f(Y_t|\Theta, \mathcal{F}_{t-1}) \Leftarrow Y_{t+1} = \Theta\Phi_t + \Sigma W_{t+1} \tag{2}$$

- Prior distribution $f(\Theta|\mathcal{F}_0)$ describes beliefs - conjugate form.

In a model with constant parameters, we use Bayesian data update

$$f(\Theta|Y_t, \mathcal{F}_{t-1}) = \frac{f(Y_t|\Theta, \mathcal{F}_{t-1})f(\Theta|\mathcal{F}_{t-1})}{\int_\Omega f(Y_t|\Theta, \mathcal{F}_{t-1})f(\Theta|\mathcal{F}_{t-1})d\Theta} \tag{3}$$

In a model with Gaussian innovations and GiW (NiG) prior the estimation has two important properties

- Exponential form of density – transforms multiplication into summation of exponents
- Quadratic (polynomial) form in the exponent conserves form when summed with other quadratic form (polynomial of same order)

in a model with Laplace innovations and conjugate prior the first property holds, while the second one fails

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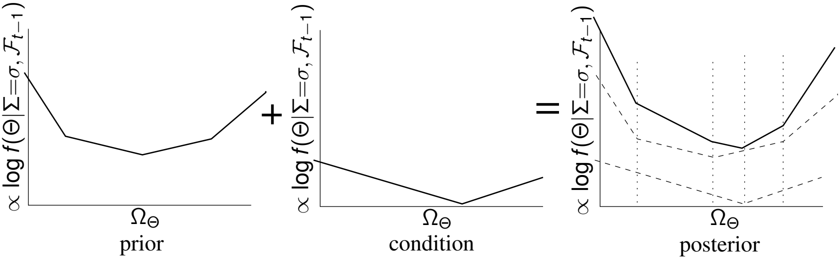


Figure: The principal of update of logarithmized conditional probability density of parameters $f(\Theta|\Sigma, \mathcal{F}_{t-1})$ in a model with Laplace noise via Bayes rule in one dimension. The logarithm transforms multiplication into addition.

and Metropolis-Hastings or

$$J_t = \sum_{P_j \in \mathcal{C}} \frac{\mathrm{vol}(P_j)\Gamma(\nu_t - n - 1)}{2^{\nu_t} n!} \sum_{k=1}^{n+1} \frac{1}{H_k^{\nu_t - n - 1}} \prod_{l=1, k \neq l}^{n+1} \frac{1}{H_l - H_k} \tag{4}$$

where H_k is value of exponent at point $P_k^0 \in \mathcal{C}$.

TABLE I

EMPIRICAL VARIANCES OF SOME ALTERNATIVE LOCATION ESTIMATORS[a]

(Sample Size 20)

tail heaviness - kurtosis →

Estimators	Distributions: Normal	10% 3σ[b]	10% 10σ[c]	Laplace	Cauchy	
Mean	1.00	1.88	11.54	2.10	12,548.0	Gauss
10% trimmed mean	1.06	1.31	1.46	1.60	7.3	
25% trimmed mean	1.20	1.41	1.47	1.33	3.1	
Median	1.50	1.70	1.80	1.37	2.9	Laplace
Gastwirth[d]	1.23	1.45	1.51	1.35	3.1	
Trimean[e]	1.15	1.37	1.48	1.43	3.9	

[a] Abstracted from Exhibit 5 in Andrews, *et al.* [3].
[b] Gaussian Mixture: $.9\Phi(1)+.1\Phi(3)$.
[c] Gaussian Mixture: $.9\Phi(1)+.1\Phi(10)$.
[d] $\tilde{\beta}=.3\beta^*(1/3)+.4\beta^*(1/2)+.3\beta^*(2/3)$, where $\beta^*(\theta)$ is the θth sample quantile.
[e] $\tilde{\beta}=1/4\beta^*(1/4)+1/2\beta^*(1/2)+1/4\beta^*(3/4)$.

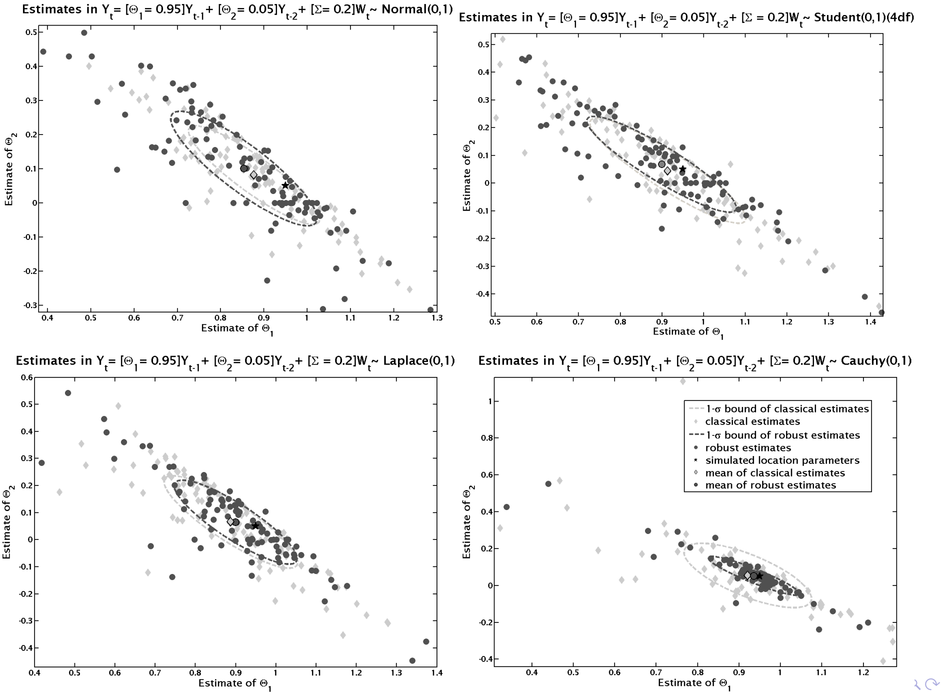
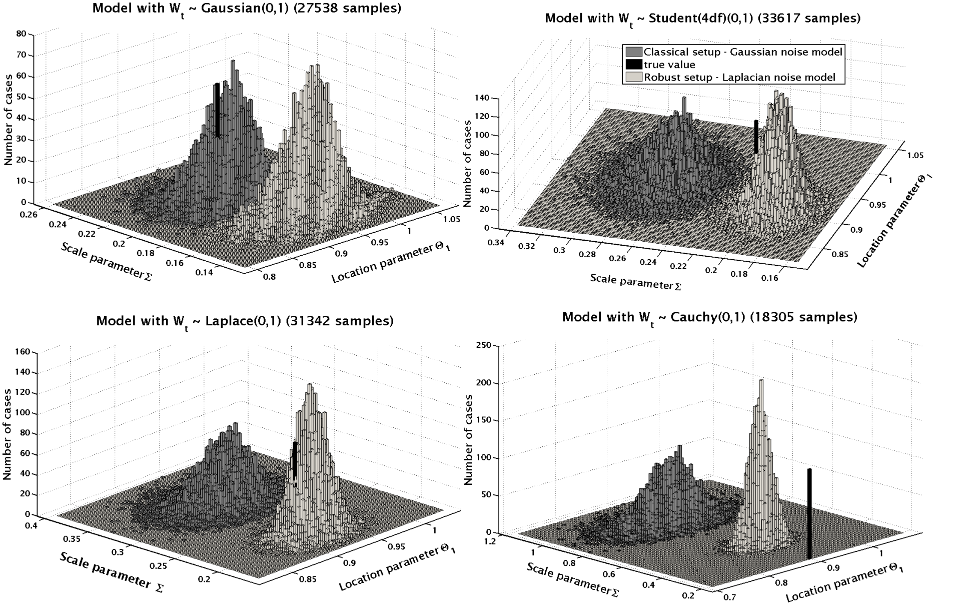

Estimates in Y_t = [Θ_1 = 0.95]Y_t-1 + [Θ_2 = 0.05]Y_t-2 + [Σ = 0.2]W_t ~ Normal(0,1)
Estimates in Y_t = [Θ_1 = 0.95]Y_t-1 + [Θ_2 = 0.05]Y_t-2 + [Σ = 0.2]W_t ~ Student(0,1)(4df)
Estimates in Y_t = [Θ_1 = 0.95]Y_t-1 + [Θ_2 = 0.05]Y_t-2 + [Σ = 0.2]W_t ~ Laplace(0,1)
Estimates in Y_t = [Θ_1 = 0.95]Y_t-1 + [Θ_2 = 0.05]Y_t-2 + [Σ = 0.2]W_t ~ Cauchy(0,1)
Estimate of Θ_1
Estimate of Θ_2
1-σ bound of classical estimates
classical estimates
1-σ bound of robust estimates
robust estimates
simulated location parameters
mean of classical estimates
mean of robust estimates

Table: Descriptive statistics of maximum likelihood estimates of location parameters in model $y_t = (\alpha_1 = 0.95)y_{t-1} + (\alpha_2 = 0.05)y_{t-2} + 0.2e_t$ for different noises and two expected models.

model\real	$\mathcal{N}(0,1)$		$t_4(0,1)$	
	$\mathrm{mean}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{cov}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{mean}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{cov}(\hat{\alpha}_1, \hat{\alpha}_2)$
Gaussian	$\begin{pmatrix} .8771 \\ .0820 \end{pmatrix}$	$\begin{pmatrix} .0229 & -.0196 \\ -.0196 & .0208 \end{pmatrix}$	$\begin{pmatrix} .9136 \\ .0448 \end{pmatrix}$	$\begin{pmatrix} .0350 & -.0322 \\ -.0322 & .0357 \end{pmatrix}$
Laplacian	$\begin{pmatrix} .8535 \\ .1011 \end{pmatrix}$	$\begin{pmatrix} .0285 & -.0247 \\ -.0247 & .0281 \end{pmatrix}$	$\begin{pmatrix} .8988 \\ .0682 \end{pmatrix}$	$\begin{pmatrix} .0321 & -.0281 \\ -.0281 & .0297 \end{pmatrix}$

model\real	$\mathcal{L}(0,1)$		$\mathcal{C}(0,1)$	
	$\mathrm{mean}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{cov}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{mean}(\hat{\alpha}_1, \hat{\alpha}_2)$	$\mathrm{cov}(\hat{\alpha}_1, \hat{\alpha}_2)$
Gaussian	$\begin{pmatrix} .8876 \\ .0661 \end{pmatrix}$	$\begin{pmatrix} .0275 & -.0247 \\ -.0247 & .0290 \end{pmatrix}$	$\begin{pmatrix} .9207 \\ .0550 \end{pmatrix}$	$\begin{pmatrix} .0236 & -.0193 \\ -.0193 & .0303 \end{pmatrix}$
Laplacian	$\begin{pmatrix} .8998 \\ .0652 \end{pmatrix}$	$\begin{pmatrix} .0232 & -.0205 \\ -.0205 & .0239 \end{pmatrix}$	$\begin{pmatrix} .9354 \\ .0532 \end{pmatrix}$	$\begin{pmatrix} .0121 & -.0106 \\ -.0106 & .0110 \end{pmatrix}$



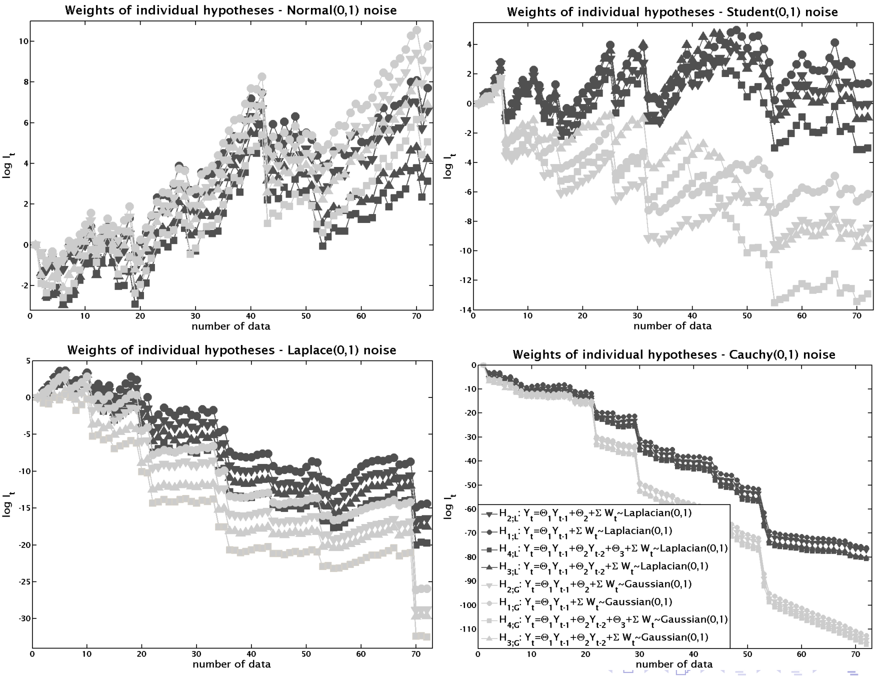
Model with W_t ~ Gaussian(0,1) (27538 samples)
Number of cases
Scale parameter Σ
Location parameter Θ_1
Model with W_t ~ Student(4df)(0,1) (33617 samples)
Classical setup - Gaussian noise model
true value
Robust setup - Laplacian noise model
Number of cases
Scale parameter Σ
Location parameter Θ_1
Model with W_t ~ Laplace(0,1) (31342 samples)
Number of cases
Scale parameter Σ
Location parameter Θ_1
Model with W_t ~ Cauchy(0,1) (18305 samples)
Number of cases
Scale parameter Σ
Location parameter Θ_1

$$
\begin{aligned}
H_1 : Y_t &= \Theta_1 Y_{t-1} + \sigma W_t \\
H_2 : Y_t &= \Theta_1 Y_{t-1} + \Theta_2 + \sigma W_t \\
H_3 : Y_t &= \Theta_1 Y_{t-1} + \Theta_2 Y_{t-2} + \sigma W_t \\
H_4 : Y_t &= \Theta_1 Y_{t-1} + \Theta_2 Y_{t-2} + \Theta_3 + \sigma W_t
\end{aligned}
$$

$$
R : Y_t = [\Theta_1 = 0.9]\, Y_{t-1} + [\Sigma = 0.2]\, W_t
$$

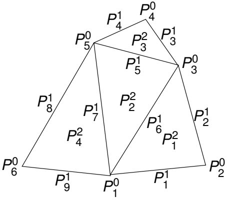
real\hypo	$H_1{+}W_t{\sim}\mathcal{L}$	$H_2{+}W_t{\sim}\mathcal{L}$	$H_3{+}W_t{\sim}\mathcal{L}$	$H_4{+}W_t{\sim}\mathcal{L}$	$H_1{+}W_t{\sim}\mathcal{N}$	$H_2{+}W_t{\sim}\mathcal{N}$	$H_3{+}W_t{\sim}\mathcal{N}$	$H_4{+}W_t{\sim}\mathcal{N}$
$\mathcal{N}^{a}(0,1)$	0.486	0.013	0.164	0.004	0.29	0.001	0.043	$5.3{\cdot}10^{-7}$
$\mathcal{N}^{b}(0,1)$	0.191	0.007	0.006	0.002	0.63	0.013	0.093	0.002
$t_4(0,1)$	0.88	0.026	0.092	0.003	$5.4{\cdot}10^{-6}$	$3.4{\cdot}10^{-9}$	$2.8{\cdot}10^{-8}$	$1.5{\cdot}10^{-11}$
$\mathcal{L}(0,1)$	0.876	0.048	0.073	0.004	$2.6{\cdot}10^{-7}$	$3{\cdot}10^{-10}$	$9.1{\cdot}10^{-11}$	$7.9{\cdot}10^{-13}$
$\mathcal{C}(0,1)$	0.87	0.096	0.031	0.004	$7.0{\cdot}10^{-18}$	$8.1{\cdot}10^{-21}$	$4.5{\cdot}10^{-20}$	$5.8{\cdot}10^{-23}$

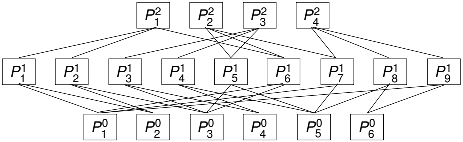
[a,b] The choice of hypotheses for data from model with Normally distributed noise depends on the chosen prior distribution. We therefore give results for a prior specified by density $f(\Lambda|\mathcal{F}_0)$ the result is given in line $\mathcal{N}^a(0,1)$ and also a result with prior specified by $f(\Lambda|\mathcal{F}_1)$ (posterior after one update) the result given in line $\mathcal{N}^b(0,1)$.



Weights of individual hypotheses - Normal(0,1) noise
Weights of individual hypotheses - Student(0,1) noise
Weights of individual hypotheses - Laplace(0,1) noise
Weights of individual hypotheses - Cauchy(0,1) noise
$\log l_t$
number of data
$H_{2;L}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2+\Sigma\ W_t$~Laplacian(0,1)
$H_{1;L}$: $Y_t=\Theta_1 Y_{t-1}+\Sigma\ W_t$~Laplacian(0,1)
$H_{4;L}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2 Y_{t-2}+\Theta_3+\Sigma\ W_t$~Laplacian(0,1)
$H_{3;L}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2 Y_{t-2}+\Sigma\ W_t$~Laplacian(0,1)
$H_{2;G}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2+\Sigma\ W_t$~Gaussian(0,1)
$H_{1;G}$: $Y_t=\Theta_1 Y_{t-1}+\Sigma\ W_t$~Gaussian(0,1)
$H_{4;G}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2 Y_{t-2}+\Theta_3+\Sigma\ W_t$~Gaussian(0,1)
$H_{3;G}$: $Y_t=\Theta_1 Y_{t-1}+\Theta_2 Y_{t-2}+\Sigma\ W_t$~Gaussian(0,1)

Complexity:





$$N_n(r,t) = \sum_{p=n-r}^{n} \binom{p}{n-r} \binom{t}{p}$$

$$\lim_{t\to\infty} N_n(r,t) < t^n \quad \forall r \in \{0,1,\ldots,n\}$$

Thank you for your attention.