

# Stochastic Robust (Anisotropy-based) Control Theory:

## New Sight, New Problems

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# AGENDA

- Introduction
  1. Disturbance attenuation problem
  2. Examples of such problems  
 $LQG$  and  $H_\infty$  optimization. Difference and commonality.
  3. Invariance theory (Schippanov's Theory)

# Anisotropy-based theory

- Optimal anisotropy-based optimization problem (1993-2005)
  1. Fundamentals of the theory: anisotropy of the signal, mean anisotropy of the sequence, physical interpretation, how to calculate.
  2. Anisotropic norm: properties, how to calculate.  
Asymptotic of anisotropic norm
  3. Anisotropy - based optimal control problem. The problem solution. Equations for optimal control design.
  4. Computation tool for control design. Gomotopy method for solving cross-coupled equations.
  5. Anisotropy-based small gain theorem. Criteria of robust stability.
  6. Anisotropic-based optimal control problem for the systems with parametric uncertainties.

- Suboptimal anisotropy- based problem. (2005 - 2012)
  1. Suboptimal anisotropy- based problem. KYP lemma for suboptimal problem. LMI methods in Anisotropic theory. Semidefinite programming in anisotropic theory.
  2. Model reduction in anisotropy - based theory.

# Application of anisotropy-based theory

- Anisotropy-based theory for descriptor systems. (2012 - ????)
  1. Anisotropy-based theory for descriptor systems.  
Analysis problem. Synthesis problem.
  2. How to calculate anisotropic norm for descriptor systems.
  3. Suboptimal problem for descriptor system.
- Adaptive anisotropy-based control
  1. How to find generating filter for concrete mean anisotropy level. Signal processing problem.
- How to extend anisotropic theory to some non linear systems. Absolute stability.

## Anisotropy-based theory with non-zero expectation of input disturbance

- Optimal anisotropy-based optimization problem
  1. Fundamentals of the theory in this case: anisotropy of the signal, mean anisotropy of the sequence, physical interpretation, how to calculate.
  2. Anisotropic norm: how to calculate.
  3. Anisotropy - based optimal control problem in this case. Some ideas how to solve a problem.

# Introduction

## Disturbance attenuation problem

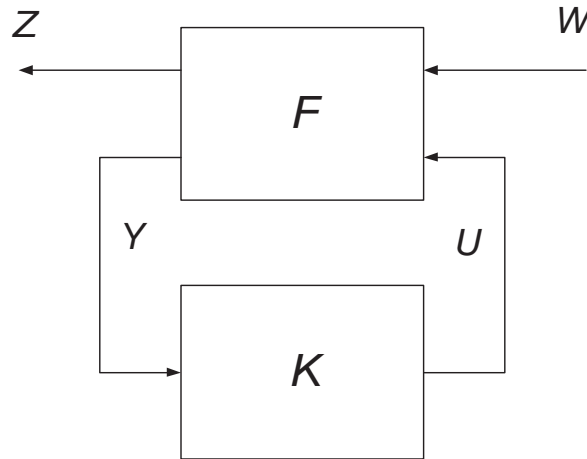


Рис. 1: Control for disturbance attenuation

$F$  is the plant,  $K$  is a controller,  $W$  and  $Z$  are input and output appropriately,  $Y$  and  $U$  are observing output and control.

Find control, which minimizes the gain from  $W$  to  $Z$ .



**Examples of such problems.**  
*LQG* and  $H_\infty$  optimization.  
**Schipanov's invariance theory**

If  $F$  and  $K$  are linear operators, there are two well known theories, which are the particular cases of theory above:  
*LQG*/ $H_2$  and  $H_\infty$  optimization.

What are the gains in those theories and which sense of those gains?

Another example: Schipanov's invariance theory (1939 - "Automation and remote control").

$Z = 0$  absolute invariance,  $Z \leq \epsilon$  invariance accurate within  $\epsilon$ .

## Standard $H_2$ optimization problem

**Standard  $H_2$  optimization problem:** Find the controller  $K$ , which

- stabilizes close loop system
- minimizes  $H_2$  norm of close loop transfer function matrix  $T_{ZW}$  from  $W$  to  $Z$ :

$$\|T_{ZW}\|_2 \rightarrow \min \quad (1)$$

Definition:

$$\|H\|_2 = \left( \text{Tr} \int_{-\pi}^{\pi} \hat{H}(\omega) (\hat{H}(\omega))^* d\omega \right)^{1/2}, \quad (2)$$

where

$$\hat{H}(\omega) \equiv \lim_{r \rightarrow 1-0} H \left( r e^{i\omega} \right), \quad \omega \in \Omega \equiv [-\pi; \pi],$$

is the angular boundary value of the generating filter  $H$ .

## Standard $H_\infty$ optimization problem

**Standard  $H_\infty$  optimization problem:** Find the controller  $K$ ,

- stabilizes close loop system
- minimizes  $H_\infty$  norm of close loop transfer function matrix  $T_{ZW}$  from  $W$  to  $Z$ :

$$\|T_{ZW}\|_\infty \rightarrow \min \quad (3)$$

**Suboptimal  $H_\infty$  control problem:**

$$\|T_{ZW}\|_\infty \leq \gamma, \quad (4)$$

где  $\gamma \geq \gamma_{opt}$ ,  $\gamma_{opt} \geq \|T_{ZW}\|_\infty$ .

For transfer function matrix  $H(z)$  the define

$$\|H\|_\infty \equiv \sup_{|z|<1} \bar{\sigma}(H(z)) = \text{ess sup}_{\omega \in \Omega} \bar{\sigma}(\hat{H}(\omega)), \quad (5)$$

where  $\bar{\sigma}(\cdot)$  is maximal singular value of matrix.

## Similarity and difference of $H_\infty$ and $H_2$ control problems

**Similarity.** The solving of both problems are based on solutions of Riccati equations, in  $H_\infty$  suboptimal control problem Riccati equation has some parameter  $\gamma$ . If  $\gamma \rightarrow \infty$  the Riccati equations for  $H_\infty$  suboptimal control problem tend to Riccati equations for  $LQG$  control problem.

**Difference. Frequency interpretation for  $H_\infty$  and  $H_2$  optimal problem for SISO systems :**  $H_\infty$  controllers are designed to minimize maximum of amplitude-frequency characteristic of closed-loop system,  $H_2$  control minimizes the average amplitude over all frequencies.

**Input signal assumptions:** Input disturbance  $W$  is to be gaussian white noise in  $LQG$  problem. Input disturbance  $W$  is quadratic integrable in  $H_\infty$  problem.

## Singularity of $H_\infty$ and $H_2$ controllers functioning if input signal assumptions are not true.

The close loop system does not work good with  $H_2$  controller in disturbance attenuation problem if the input signal is «far from» white noise.

The close loop system with  $H_\infty$  controller is very conservative (the great amount of control needed) if the input signal is «closed enough» to gaussian white noise.

## Convergence (trade-off) between $H_\infty$ and $H_2$ theories

### Capability of common (joint) theory construction

- Optimal ( suboptimal)  $H_\infty$  controllers are not unique. It means we can propose once more performance criterion .
- Natural choice for the new performance criterion is  $H_2$  norm of close loop transfer function matrix.

1. Minimization of close loop system  $H_2$  norm with constraints on  $H_\infty$  norm.

Bernstein D.A., Haddad W.M. *LQG* Control with an  $H_\infty$  Performance Bound: A Riccati Equation Approach. //IEEE Transactions on Automatic Control, AC-34, N 3, 1989.

2. Minimization of close loop system  $H_\infty$  norm with upper bound  $H_2$  norm minimization.

Mustafa D., Glover K. Minimum Entropy  $H_\infty$ -Control. Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin etc., 1991.

## $H_\infty$ optimization problem with minimization of $H_\infty$ entropy

On the set of  $H_\infty$  suboptimal controllers to find controller which minimizes  $H_\infty$  entropy functional

$$J(\gamma, F) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det (I_m - \gamma^{-2} (F(j\omega))^* F(j\omega))| d\omega,$$

$\gamma$  is the number that bounds close loop transfer function  $H_\infty$  norm for stable close loop system  $F(s)$ .

The minimization of  $H_\infty$  entropy of the system  $F(s)$  is equivalent of the minimization of upper bound of  $H_2$  norm of  $F(s)$ .

- Designed controller is unique .
- $H_\infty$  control problem with  $H_\infty$  entropy minimization is equivalent to risk sensitivity problem.

**Class of control systems anisotropy theory was done for.  
Mathematical models for investigation**

$$\left\{ \begin{array}{l} x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \\ z_k = C_1 x_k + D_{11} w_k + D_{12} u_k \\ y_k = C_2 x_k + D_{21} w_k \end{array} \right. , \quad -\infty < k < +\infty ,$$

(6)

where  $A$ ,  $C_i$ ,  $B_j$  и  $D_{ij}$  are appropriate dimension constant matrixes. System  $F(z)$ , and its subsystems  $F(z)_{ij}$  have following state space realizations:

$$F \sim \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] ,$$

(7)

$$F_{ij} \sim \left[ \begin{array}{cc} A & B_j \\ C_i & D_{ij} \end{array} \right] , \quad 1 \leq i, j \leq 2$$

(8)



# Anisotropy theory background

1. **Fundamentals of the theory: anisotropy of the signal, mean anisotropy of the sequence, physical interpretation**  
How to calculate

**Definition 1** *The relative entropy (Kullback-Leibler distance)  $D(f \parallel g)$  between two densities  $f(x)$  and  $g(x)$  is defined by*

$$D(f \parallel g) = \int f(x) \log \frac{f(x)}{g(x)} dx. \quad (9)$$

*$D(f \parallel g)$  is finite, if support set of  $f(x)$  is contained in the support set of  $g(x)$ . It is true that  $0 \log \frac{0}{0} = 0$ .*

$$D(f \parallel g) \geq 0$$

with equality iff  $f = g$  almost everywhere.

**Definition 2** *Let  $X$  and  $Y$  are two random variable with joint distribution function of probability density  $f(x, y)$  and probability density functions  $f(x)$   $f(y)$  appropriately. The mutual information  $I(X; Y)$  is defined as*

$$I(X; Y) = \int \log f(x, y) \frac{f(x, y)}{f(x)f(y)} dx dy. \quad (10)$$

## Definition of anisotropy of the random vector

Denote by  $\mathbb{L}_2^m$  the class  $\mathbb{R}^m$ -dimension absolutely continuously distributed random vectors  $W$  with values in  $R^m$  satisfying  $\mathbf{E}|W|^2 < \infty$ .

For any  $\lambda > 0$  denote as  $p_{m,\lambda}$  the probability density function (pdf) on  $\mathbb{R}^m$  of gaussian signal with zero mean and scalar covariance matrix  $\lambda I_m$

$$p_{m,\lambda}(w) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{|w|^2}{2\lambda}\right), \quad w \in \mathbb{R}^m. \quad (11)$$

For any  $W \in \mathbb{L}_2^m$  with pdf  $f : \mathbb{R}^m \rightarrow \mathbb{R}_+$  the relative entropy of  $W \in \mathbb{L}_2^m$  according to (11) has the following view

$$D(f||p_{m,\lambda}) = \mathbf{E}_f \ln \frac{f(W)}{p_{m,\lambda}(W)} = -h(W) + \frac{m}{2} \ln(2\pi\lambda) + \frac{\mathbf{E}|W|^2}{2\lambda}, \quad (12)$$

where

$$h(W) = -\mathbf{E} \ln f(W) = -\int_{\mathbb{R}^m} f(w) \ln f(w) dw \quad (13)$$

is differential entropy of random vector  $W$

## Definition of anisotropy of the random vector (continuation)

**Definition 3** *The anisotropy  $\mathbf{A}(W)$  of random vector  $W \in \mathbb{L}_2^m$  is defined as minimal relative entropy of its pdf from gaussian distribution  $\mathbb{R}^m$  with zero mean and scalar covariance matrix*

$$\mathbf{A}(W) = \min_{\lambda > 0} D(f \| p_{m,\lambda}). \quad (14)$$

Direct calculation shows, that minimum in (12) over  $\lambda > 0$  is obtained if  $\lambda = \mathbf{E}|W|^2/m$ , so

$$\mathbf{A}(W) = \min_{\lambda > 0} D(f \| p_{m,\lambda}) = \frac{m}{2} \ln \left( \frac{2\pi e}{m} \mathbf{E}|W|^2 \right) - h(W). \quad (15)$$

## Properties of random vector anisotropy

Denote by  $\mathbb{G}^m(\Sigma)$  the class of  $\mathbb{R}^m$ -valued gaussian disturbances random vectors  $W$  with  $\mathbf{E}W = 0$  and nonsingular covariance matrix  $\mathbf{cov}(W) = \Sigma$ , so that the corresponding pdf is

$$p(w) = (2\pi)^{-m/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2} \|w\|_{\Sigma^{-1}}^2\right),$$

$\|x\|_Q = \sqrt{x^\top Q x}$  denotes the norm of a vector  $x$ , induced by a positive definite symmetric matrix  $Q > 0$ .

### Lemma 1

(a) For any positive definite matrix  $\Sigma \in \mathbb{R}^{m \times m}$ ,

$$\min_W \left\{ \mathbf{A}(W) : W \in \mathbb{L}_2^m, \mathbf{E}(WW^\top) = \Sigma \right\} = -\frac{1}{2} \ln \det \frac{m\Sigma}{\text{Trace } \Sigma}, \quad (16)$$

and the minimum is attained only for  $W \in \mathbb{G}^m(\Sigma)$ ;

(b) For any  $W \in \mathbb{L}_2^m$ ,  $\mathbf{A}(W) \geq 0$ . Moreover  $\mathbf{A}(W) = 0$  iff  $W \in \mathbb{G}^m(\lambda I_m)$

## Mean anisotropy of random sequences

Let  $W \in \mathbb{L}_2^m$  be partitioned into subvectors  $w_1, \dots, w_r$  of dimensions  $m_1, \dots, m_r$ , e.g.  $m_1 + \dots + m_r = m$

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}. \quad (17)$$

For any  $1 \leq s \leq t \leq r$ , denote by  $W_{s:t} = (w_k)_{s \leq k \leq t}$  the  $(m_s + \dots + m_t)$ -dimensional subvector of  $W$  (17), obtained by "stacking"  $w_s, \dots, w_t$ .

**Definition 4** *The mean anisotropy of sequence  $W$  is defined as:*

$$\overline{\mathbf{A}}(W) = \lim_{N \rightarrow +\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}. \quad (18)$$

## Mean anisotropy of gaussian random sequences

Let  $V \equiv (v_k)_{-\infty < k < +\infty} \in \mathbb{G}^m(I)$ ,  $W \equiv (w_k)_{-\infty < k < +\infty} \equiv GV$ ,

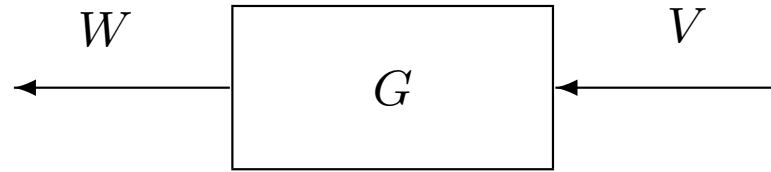


Рис. 2:

The generating filter  $G \in H_2^{m \times m}$  is identified with its transfer function

$$G(z) \equiv \sum_{k=0}^{+\infty} g_k z^k, \text{ where } g_k \in \mathbb{R}^{m \times m}, \quad k \geq 0 \text{ is input-impulse response.}$$

**Theorem 1** *The mean anisotropy (18) can be representable as*

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \det \left( \frac{m}{\|G\|_2^2} \hat{G}(\omega) \left( \hat{G}(\omega) \right)^* \right) d\omega. \quad (19)$$

## Properties of gaussian sequence mean anisotropy

- $\overline{\mathbf{A}}(W) > 0$  if  $\text{rank } \widehat{G}(\omega) = m$  for almost all  $\omega \in [-\pi, \pi)$ ,
- $\overline{\mathbf{A}}(W) = +\infty$  if  $\widehat{G}$  - not maximum rank,
- $\overline{\mathbf{A}}(W) = 0$  if there is such number  $\alpha > 0$  что  $\widehat{G}(\omega)\widehat{G}^*(\omega) = \alpha I_m, \quad -\pi \leq \omega < \pi$ .



## Calculation of mean anisotropy in state space

Let state space representation of generating filter  $G \in H_2^{m \times m}$  be

$$\begin{cases} x_{k+1} &= Ax_k + Bv_k \\ w_k &= Cx_k + Dv_k \end{cases}, \quad -\infty < k < +\infty, \quad (20)$$

where  $A, B, C, D$  are matrices of appropriate dimension. The matrix  $\rho(A) < 1$  is assumed to be asymptotically stable (with spectral radius  $\rho(a) < 1$ ) and  $D$  nonsingular.

Associate with the filter  $G$  the Riccati equation in the matrix  $R \in \mathbb{R}^{n \times n}$

$$R = ARA^\top + BB^\top - \Lambda\Theta\Lambda^\top, \quad (21)$$

$$\Lambda \doteq (ARC^\top + BD^\top)\Theta^{-1}, \quad (22)$$

$$\Theta \doteq CRC^\top + DD^\top. \quad (23)$$

A solution  $R$  of equation (21)–(23) is said to be *admissible* if  $R$  is symmetric and positive semidefinite and matrix  $A - \Lambda C$  is asymptotically stable.

## Calculation of mean anisotropy in state space

The equation (21)–(23) can be written in a form

$$\begin{aligned} &ARA^\top - R - (ARC^\top + BD^\top) \\ &\quad \times (CRC^\top + DD^\top)^{-1}(CRA^\top + DB^\top) + BB^\top = 0. \end{aligned} \quad (24)$$

**Theorem 2** *Let a generating filter  $G \in H_2^{m \times m}$  have state-space realization (20) with  $A$  asymptotically stable and  $D$  nonsingular. Then the mean anisotropy (19) of the sequence  $W = GV$  is*

$$\bar{\mathbf{A}}(G) = -\frac{1}{2} \ln \det \left( \frac{m \Theta}{\text{Trace}(CPC^\top + DD^\top)} \right), \quad (25)$$

where  $\Theta = CRC^\top + DD^\top$ ,  $R$  is admissible Riccati equation (21)–(23), and  $P$  is controllability gramian of the filter satisfying Lyapunov equation

$$P = APA^\top + BB^\top. \quad (26)$$

## Algorithm for mean anisotropy calculation

- The Riccati equation (21)–(23) or (24):

$$ARA^{\top} - R - (ARC^{\top} + BD^{\top}) \\ \times (CRC^{\top} + DD^{\top})^{-1}(CRA^{\top} + DB^{\top}) + BB^{\top} = 0.$$

is solved, and  $R$  and  $\Theta = CRC^{\top} + DD^{\top}$  is found

- Lyapunov equation

$$P = APA^{\top} + BB^{\top}.$$

is solved.

- The mean anisotropy is calculated by formula

$$\overline{\mathbf{A}}(G) = -\frac{1}{2} \ln \det \left( \frac{m \Theta}{\text{Trace}(CPC^{\top} + DD^{\top})} \right).$$

**2. Anisotropic norm: properties, how to calculate.  
Asymptotic of anisotropic norm.**

## Anisotropic norm of linear time invariant systems

Let  $F(z) \in H_\infty^{p \times m}$  be linear time invariant system and  $Z = FW$ , e.g.  $F(z)$  is analytic in open unit ball and has finite  $H_\infty$  norm  $\|F\|_\infty = \sup_{|z| < 1} \bar{\sigma}(F(z)) =$

$\text{ess sup}_{-\pi \leq \omega \leq \pi} \bar{\sigma}(\widehat{F}(\omega))$ , where  $\bar{\sigma}(\cdot)$  is maximum singular value of  $F(z)$ .

**Definition 5** For given  $a \geq 0$ ,  $a$ -anisotropic norm of the system  $F$  is defined as

$$\|F\|_a = \sup_G \{ \|FG\|_2 / \|G\|_2 : G \in \mathbf{G}_a \}, \quad (27)$$

$$\mathbf{G}_a = \{ G \in H_2^{m \times m} : \bar{\mathbf{A}}(G) \leq a \} \quad (28)$$

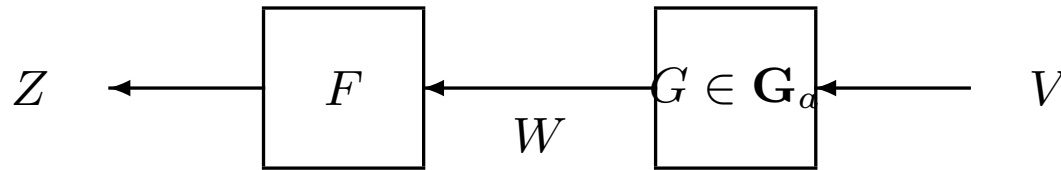


Рис. 3:

## Properties of anisotropic norm for linear system

For any fixed system  $F \in H_\infty^{p \times m}$ , its  $a$ -anisotropic norm (53) is nondecreasing continuous function of  $a \geq 0$  satisfying

$$\frac{1}{\sqrt{m}} \|F\|_2 = \|F\|_0 \leq \|F\|_a \leq \lim_{a \rightarrow +\infty} \|F\|_a = \|F\|_\infty. \quad (29)$$

By (29), computing the norm  $\|F\|_a$  is only of interest if  $a > 0$  and

$$\|F\|_2 < \sqrt{m} \|F\|_\infty \quad (30)$$

(there is a particular interest if  $\|F\|_\infty \gg \|F\|_2/\sqrt{m}$ ). This equality is not true iff,  $F$  is an inner (inner system) up to a nonzero constant multiplier  $\lambda > 0$  such that  $(\widehat{F}(\omega))^* \widehat{F}(\omega) = \lambda I_m$  for almost all  $\omega \in [-\pi, \pi)$ . For nonzero system  $F \in H_\infty^{p \times m}$ , the inequality  $p < m$  implies (30).

## Anisotropic norm of linear system

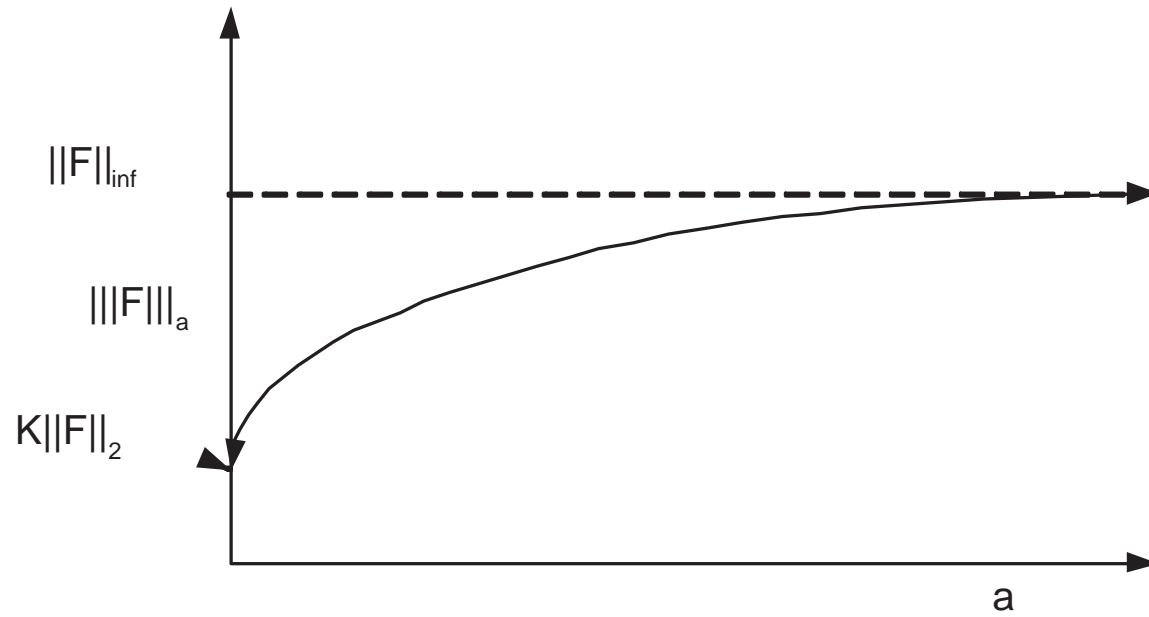


Рис. 4: Changes of anisotropic norm

$$K = \frac{1}{\sqrt{m}}$$

## Asymptotic behavior of $a$ - anisotropic norm

$$\|F\|_a - \frac{\|F\|_2}{\sqrt{m}} \sim \frac{\sqrt{\|F\|_4^4/m - (\|F\|_2^2/m)^2}}{\|F\|_2} \sqrt{a} \quad \text{if } a \rightarrow 0+, \quad (31)$$

$$\|F\|_\infty - \|F\|_a \sim \frac{1}{2} \|F\|_\infty \exp\left(-\frac{2}{m} (J(\|F\|_\infty) + a)\right) \quad \text{if } a \rightarrow +\infty \quad (32)$$

For any positive integer  $k$ , the norm of the system  $F \in H_\infty^{p \times m}$  in Hardy space  $H_{2k}^{p \times m}$  is defined as

$$\|F\|_{2k} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Trace} \left( (\widehat{F}(\omega))^* \widehat{F}(\omega) \right)^k d\omega \right)^{1/(2k)}$$

(particularly, for  $k = 1$ , it gives  $H_2$ -norm).

$$J(\gamma, F) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det (I_m - \gamma^{-2} (F(j\omega))^* F(j\omega))| d\omega,$$



## Pseudo multiplicative property of anisotropic norm

The ring property of  $H_\infty$ -norm , (sub multiplicative property)

$$\|FG\|_\infty \leq \|F\|_\infty \|G\|_\infty$$

is not true for anisotropic norm  $\|\cdot\|_a$ .

But there is the analog of ring property.

**Theorem 3** For any  $a \geq 0$  and any systems  $F \in H_\infty^{p \times m}$  u  $G \in H_\infty^{m \times m}$ ,

$$\|FG\|_a \leq \|F\|_b \|G\|_a \quad (33)$$

zde

$$b = a + \overline{\mathbf{A}}(G) + m \ln (\sqrt{m} \|G\|_a / \|G\|_2). \quad (34)$$

**Corollary 1** ANISOTROPIC-BASED SMALL GAIN THEOREM

## How to calculate the anisotropic norm in state space

Let system  $F$  has the following state space representation

$$F = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

It is well known fact for calculation of  $\|F\|_2$  norm of the system  $F$  it is necessary to solve Lyapunov equation.

It is well known fact for calculation of  $\|F\|_\infty$  norm of the system  $F$  it is necessary to solve Riccati equation (Bounded real lemma).

As far as anisotropic norm  $\|F\|_a$  of the system lies "between" normalized  $\|F\|_2$  and  $\|F\|_\infty$  norms, it natural to propose that we have to use Lyapunov and Riccati equation for anisotropic norm calculation. It is really true, but for the calculation algorithm we have to add some special time equation.

## How to calculate the anisotropic norm in state space II

Anisotropic norm is calculated by the formula

$$\|F\|_a = \left( \frac{1}{q} \left( 1 - \frac{m}{\text{Trace}(LPL^\top + \Sigma)} \right) \right)^{1/2}.$$

$q, P, L, \Sigma$  are unknown parameters. They can be calculated by solving coupled equations: (35) is a Riccati equation, (36) is a Lyapunov equation, (37) is a special time equation

$$\begin{aligned} R &= A^\top R A + q C^\top C + L^\top \Sigma^{-1} L, \\ L &= \Sigma (B^\top R A + q D^\top C), \\ \Sigma &= (I_m - B^\top R B - q D^\top D)^{-1}. \end{aligned} \tag{35}$$

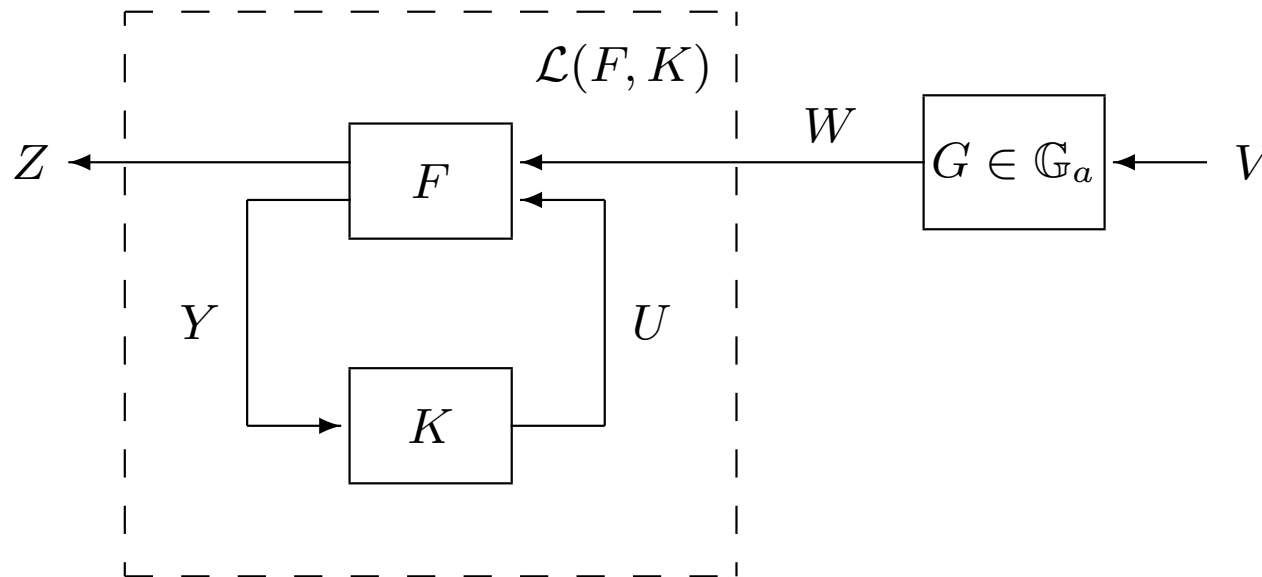
$$P = (A + B L) P (A + B L)^\top + B \Sigma B^\top, \tag{36}$$

$$a = -\frac{1}{2} \ln \det \left( \frac{m \Sigma}{\text{Trace}(L P L^\top + \Sigma)} \right). \tag{37}$$

## Anisotropy-based control design problem

Let  $W$  be generated from  $m_1$ -dimensional gaussian white noise  $V$  with zero expectation and unit covariance matrix by unknown generating filter  $G$  from

$$\mathbb{G}_a \equiv \{G \in H_2^{m_1 \times m_1} : \bar{A}(G) \leq a\} . \quad (38)$$



### 3. Anisotropic-based optimization problem:

**Problem 1** For given system  $F$  and mean anisotropy level  $a \geq 0$  of input disturbance  $W$  find the controller  $K \in \mathcal{K}$ , that minimizes the  $a$ -anisotropic norm of closed loop system  $\mathcal{F}_l(F, K)$ :

$$\|\mathcal{F}_l(F, K)\|_a \equiv \sup \left\{ \frac{\|\mathcal{F}_l(F, K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \rightarrow \inf, \quad K \in \mathcal{K}. \quad (39)$$

Let us note if  $a = 0$ , the above problem 3 is coincided with standard  $H_2$  optimization problem (Kolmogorov - Wiener-Hopf-Kalman optimization problem).

## Solution of anisotropic-based design problem

The solution of the problem is reduced to the solving of three algebraic matrix Riccati equations, Lyapynov equation and one algebraic equation of special type. If  $a = 0$  the four matrix equations turn into well known two Riccati equations from Kalman theory and the equation of special type cancels.

## 4. How to find the solution by computer? Vladimirov's Package

Crossed- coupled three matrix algebraic Riccati equations, Lyapunov equation and special type equation have been solving by homotopy method. We reduced the solution of the algebraic system to differential equation system. The anisotropy level was the independent variable in those differential systems. The initial conditions were the solutions of the problem if  $\alpha = 0$ , the *LQG* problem.

I.G. Vladimirov create the application package (software kit) for MathLab and programmed it.

## 5. ANISOTROPIC-BASED SMALL GAIN THEOREM

The ring property of  $H_\infty$ -norm , (sub multiplicative property)

$$\|FG\|_\infty \leq \|F\|_\infty \|G\|_\infty$$

is not true for anisotropic norm  $\|\cdot\|_a$ .

But there is the analog of ring property.

**Theorem 4** For any  $a \geq 0$  and any systems  $F \in H_\infty^{p \times m}$  u  $G \in H_\infty^{m \times m}$ ,

$$\|FG\|_a \leq \|F\|_b \|G\|_a \quad (40)$$

zde

$$b = a + \overline{\mathbf{A}}(G) + m \ln (\sqrt{m} \|G\|_a / \|G\|_2). \quad (41)$$

**Corollary 2** ANISOTROPIC-BASED SMALL GAIN THEOREM



## Robust stability in anisotropic theory

Let  $P$  be the object with follow description

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (42)$$

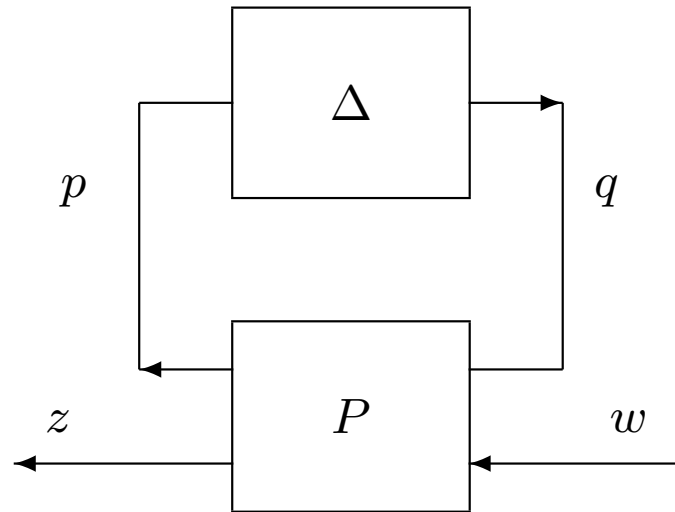


Рис. 5:  $P$ - $\Delta$  конфигурация.

**Theorem 5** Consider  $\mathcal{F}_u(P, \Delta)$ , where  $\Delta : l_2 \rightarrow l_2$  and  $P : l_2 \rightarrow l_2$  are causal linear systems.

- Let  $P$  be stable and

$$\|P_{11}\|_c < \epsilon^{-1}, \text{ where } c = a + m \ln \frac{\epsilon}{\operatorname{ess\,inf}_{-\pi \leq \omega \leq \pi} \underline{\sigma}(\Delta(j\omega))}, \quad (43)$$

$\underline{\sigma}(\Delta) = \sqrt{\lambda_{\min}(\Delta^* \Delta)}$  – minimum singular value of  $\Delta$ ,  $\epsilon > 0$ .

- Let

$$a = -\frac{1}{2} \ln \det \frac{m\Sigma}{\operatorname{tr} \Sigma} - m \ln \frac{\epsilon}{\operatorname{ess\,sup} \underline{\sigma}(\Delta(j\omega))},$$

where  $\Sigma = (I_m - qP_{11}^* P_{11})^{-1}$ , and parameter  $q \in [0, \|P_{11}\|_\infty^{-2})$  satisfies inequality

$$\operatorname{tr} \left[ (I_m - \epsilon^2 P_{11}^* P_{11}) (I_m - qP_{11}^* P_{11})^{-1} \right] \leq 0. \quad (44)$$

Then for all  $\Delta \in D_a(\epsilon)$  close-loop system  $\mathcal{F}_u(P, \Delta)$  is internal stable.

## 6. Anisotropic-based optimal control problem for the systems with parametric uncertainties.

**Problem 2** For system  $F$ , given by

$$\begin{cases} x_{k+1} &= (A + F_1\Omega_k E_1)x_k + (B_0 + F_2\Phi_k E_2)w_k + (B_2 + F_3\Psi_k E_3)u_k, \\ z_k &= C_1x_k + D_{12}u_k, \\ y_k &= C_2x_k + D_{21}w_k, \end{cases} \quad (45)$$

where  $\Omega_k, \Phi_k, \Psi_k$  are unknown with conditions:

$$\Omega_k^\top \Omega_k \leq I, \quad \Phi_k^\top \Phi_k \leq I, \quad \Psi_k^\top \Psi_k \leq I, \quad -\infty < k < +\infty, \quad (46)$$

and for given level of mean anisotropy to find the controller, that minimized

$$J_0(K) = \sup_{\Omega_k, \Phi_k, \Psi_k} \|\mathcal{F}_l(F, K)\|_a. \quad (47)$$

## Solution of anisotropic-based design problem with parametric uncertainties

The solution of the problem is reduced to the solving of four algebraic matrix Riccati equations, Lyapunov equation and one algebraic equation of special type.

## Application of anisotropy-based theory

- Suboptimal anisotropy-based problem. Anisotropy KYP lemma. Design method with LMI and semi-definite programming technic.
- Anisotropy-based theory for descriptor systems. Analysis and synthesis problems.
- Adaptive anisotropy control
- Anisotropy analysis inverse problem.

## Anisotropic-based suboptimization problem

Let  $F$  be describe by

$$\begin{cases} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k \\ y_k &= C_2 x_k + D_{21} w_k \end{cases}, \quad -\infty < k < +\infty, \quad (48)$$

**Problem 3** For given system  $F$  and mean anisotropy level  $a \geq 0$  of input disturbance  $W$  find the controller  $K \in \mathcal{K}$ , that provides the  $a$ -anisotropic norm of closed loop system  $\mathcal{F}_l(F, K)$  inequality:

$$\|\mathcal{F}_l(F, K)\|_a \equiv \sup \left\{ \frac{\|\mathcal{F}_l(F, K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \leq \gamma, \quad K \in \mathcal{K}. \quad (49)$$

## Anisotropic-based optimal problem for descriptor systems

$$\begin{cases} Ex(k+1) = Ax(k) + B_1w(k) + B_2u(k) \\ z(k) = C_1x(k) + D_{11}w(k) + D_{12}u(k) \\ y(k) = C_2x(k) + D_{21}w(k) + D_{22}u(k) \end{cases} \quad (50)$$

$\text{rank}(E) = r < n$ .

**Problem 4** For given system (52) and mean anisotropy level  $a \geq 0$  of  $W$  find  $K$  minimizing  $a$ - anisotropy norm of closed loop system :

$$\|\|\mathcal{F}_l(F, K)\|\|_a = \sup \left\{ \frac{\|\mathcal{F}_l(F, K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \rightarrow \inf, \quad (51)$$

$\mathcal{F}_l(F, K)$  is low linear fractional transformation.

## Anisotropic-based suboptimal problem for descriptor system

$$\begin{cases} Ex(k+1) = Ax(k) + B_1w(k) + B_2u(k) \\ z(k) = C_1x(k) + D_{11}w(k) + D_{12}u(k) \\ y(k) = C_2x(k) + D_{21}w(k) + D_{22}u(k) \end{cases} \quad (52)$$

$\text{rank}(E) = r < n$ .

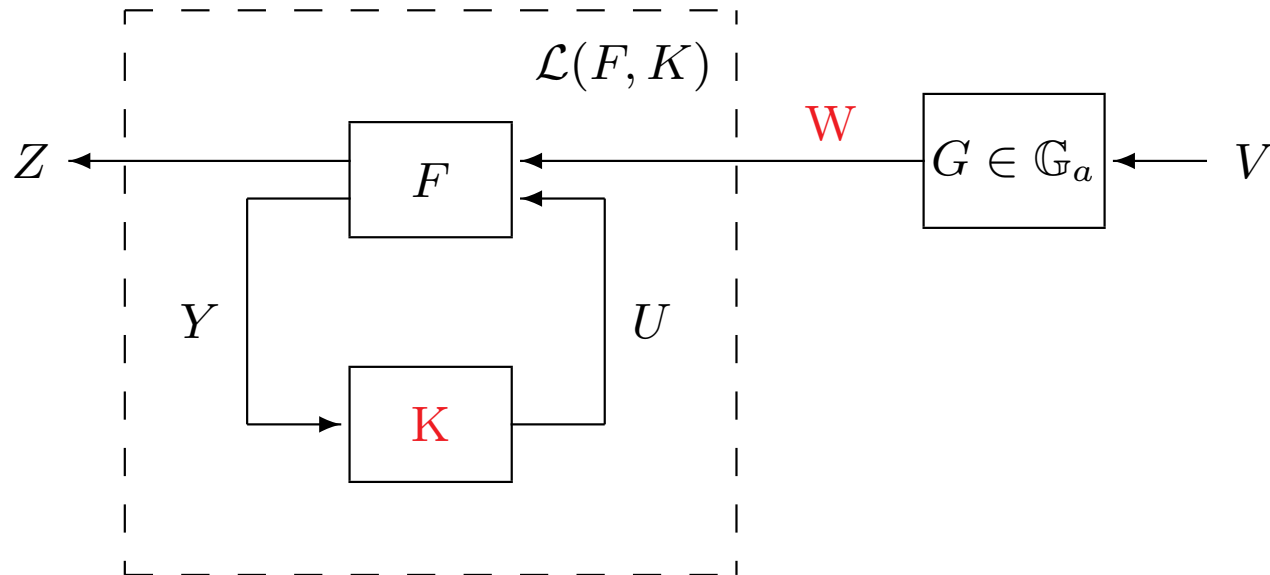
**Problem 5** For given system (52) and mean anisotropy level  $a \geq 0$  of  $W$  find  $K$  that provides  $a$ - anisotropy norm of closed loop system inequality:

$$\|\|\mathcal{F}_l(F, K)\|\|_a = \sup \left\{ \frac{\|\mathcal{F}_l(F, K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \leq \gamma, \quad (53)$$

$\mathcal{F}_l(F, K)$  is low linear fractional transformation.



# Adaptive anisotropy control



- Identification of the input disturbance parameters.
- Calculation of mean anisotropy.
- Tuning the control parameters in according to new value of mean anisotropy.

## Anisotropy analysis inverse problem

### Creation of stochastic sequence with given property

**Problem 6** *Let a level of mean anisotropy  $a$  of sequence  $\{w_k\}$  be given. Sequence  $\{w_k\}$  is received from white noise by filter*

$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{cases} \quad (54)$$

*where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{m \times n}$ ,  $D \in \mathbb{R}^{m \times m}$ .*

*Matrix  $A$  is stable and  $D$  is not singular, e.g.  $\rho(A) < 1$ ,  $\det D \neq 0$ .*

*Find the matrixes  $A, B, C, D$ .*

# Non-zero input disturbance expectation

Let pdf of input disturbances  $W$  of the system  $F$  be

$$f(x) = ((2\pi)^m |S|)^{-1/2} \exp\left(-\frac{1}{2}(x - \mu)^T S^{-1}(x - \mu)\right), \quad x \in \mathbb{R}^m.$$

For any  $\lambda > 0$  denote as  $p_{m,\lambda}$  the probability density function (pdf) on  $\mathbb{R}^m$  of gaussian signal with zero mean and scalar covariance matrix  $\lambda I_m$

$$p_{m,\lambda}(w) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{|w|^2}{2\lambda}\right), \quad w \in \mathbb{R}^m.$$

The definitions of anisotropy of the signals and mean anisotropy of the sequences will be the same as we defined above:

$$\mathbf{A}_\mu(W) \triangleq \min_{\lambda > 0} \mathbf{E}_f \ln \frac{f(x)}{p_{m,\lambda}(x)}$$

$$\overline{\mathbf{A}}_\mu(W) \triangleq \lim_{N \rightarrow \infty} \frac{\mathbf{A}_\mu(W_{0:N-1})}{N},$$

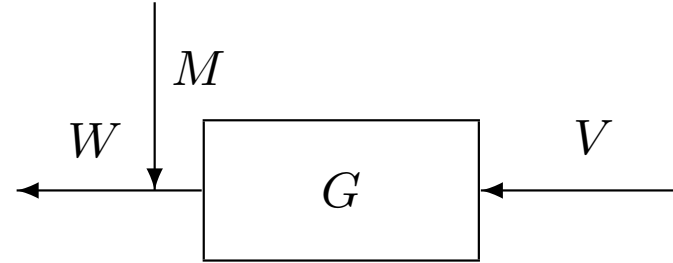
## Comparisons of the results



$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{cases}$$

$$\mathbf{A}(W) = -\frac{1}{2} \ln \det \frac{m\Sigma}{\text{tr } \Sigma}.$$

$$\bar{\mathbf{A}}(W) = -\frac{1}{2} \ln \det \left( \frac{m(\Sigma + \Xi)}{\text{tr } \Sigma} \right)$$



$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w'_k = Cx_k + Dv_k + \mu, \end{cases}$$

$$\mathbf{A}(W) = -\frac{1}{2} \ln \det \frac{m\Sigma}{\text{tr } \Sigma + |\mu|^2}.$$

$$\bar{\mathbf{A}}(W) = -\frac{1}{2} \ln \det \left( \frac{m(\Sigma + \Xi)}{\text{tr } \Sigma + |\mu|^2} \right),$$

## Anisotropic norm

Anisotropic norm for stable system  $F$  is calculated by

$$\|F\|_{a,\mu} = \sup_{\mu, q: \mathcal{A}(\mu, q) \leq a} \mathcal{N}(\mu, q), \quad \text{where}$$

$$\mathcal{N}(\mu, q) = \left( \frac{1}{q} \left( 1 - \frac{m + |X\mu|^2 - q|Y\mu|^2}{\text{tr}(LPL^T + \Sigma) + |X\mu|^2} \right) \right)^{1/2}, \quad q \in [0; \|F\|_\infty^{-2}),$$

$$X = I_m + L(I_n - (A_{cl} + B_{cl}L))^{-1}B_{cl},$$

and

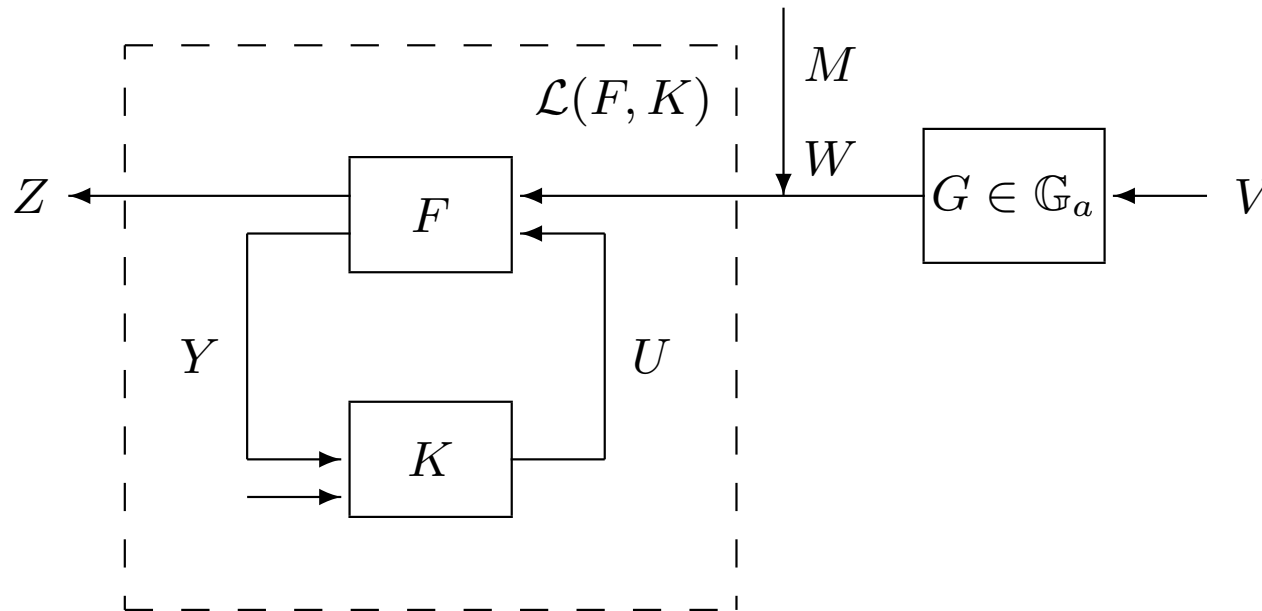
$$Y = D_{cl} + (C_{cl} + D_{cl}L)(I_n - (A_{cl} + B_{cl}L))^{-1}B_{cl}.$$

Reminder:

$$\|F\|_a = \left( \frac{1}{q} \left( 1 - \frac{m}{\text{Trace}(LPL^\top + \Sigma)} \right) \right)^{1/2}.$$

$q, P, L, \Sigma$  are unknown parameters.

## Anisotropy-based control design problem with non-zero mean of input disturbance



Now we can solve this problem under special conditions.

# Future investigations for such systems

- Suboptimal anisotropy-based problem. Anisotropy KYP lemma. Design method with LMI and semi-definite programming technic.
- Anisotropy-based theory for descriptor systems. Analysis and synthesis problems.
- Adaptive anisotropy control

# VERY DIFFICULT PROBLEM

How to extend anisotropic theory  
for continue - time systems.

Thank you very much for your attention  
and for your patience.