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On the control of distributed parameter systems using a multidimensional systems setting

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Abstract

The unique characteristic of a repetitive process is a series of sweeps, termed passes, through a set of dynamics defined over a finite duration with resetting before the start of the each new one. On each pass an output, termed the pass profile is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This leads to the possibility that the output, i.e. the sequence of pass profiles, will contain oscillations which increase in amplitude in the pass-to-pass direction. Such behavior cannot be controlled by standard linear systems approach and instead they must be treated as a multidimensional system, i.e. information propagation in more than one independent direction. Physical examples of such processes include long-wall coal cutting and metal rolling. In this paper, stability analysis and control systems design algorithms are developed for a model where a plane, or rectangle, of information is propagated in the pass-to-pass direction. The possible use of these in the control of distributed parameter systems is then described using a fourth-order wavefront equation.

Keywords: Distributed parameter systems; Modelling; Control law design

1. Introduction

Multidimensional (or nD) systems propagate information in $n>1$ independent directions and arise in many areas of, in particular, circuits, and image/signal processing. In the case of linear dynamics, this means that a transfer function description is a function of n indeterminates and this alone is a source of difficulty in terms of onward systems related analysis. For example, for functions of more than one indeterminate the fundamental tool of primeness which is at the heart of the polynomial/transfer-function approach to controllability/observability/minimality analysis (and many other problems) of standard (termed 1D here) linear systems is no longer a single concept.

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The case of discrete linear systems recursive in the upper right quadrant $(i,j): i \geqslant 0,\ j \geqslant 0$ (where i and j denote the directions of information propagation) of the 2D plane has been the subject of much research effort over the years using, in the main, the well known Roesser [1] and Fornasini Marchesini [2] state-space models. More recently, productive research has been reported on robust control using a variety of approaches—see, for example, [3,4].

In their basic form, the unique characteristic of a repetitive process (also termed a multipass process in the early literature) can be illustrated by considering machining operations where the material or workpiece involved is processed by a series of sweeps, or passes, of the processing tool. Assuming the pass length $\alpha < +\infty$ to be constant, the output vector, or pass profile, $y_k(p)$, $p = 0, 1, \ldots, (\alpha - 1)$ (p being the independent spatial or temporal variable), generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(p)$, $p = 0, 1, \ldots, (\alpha - 1)$, $k = 0, 1, \ldots$. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction, i.e. in the collection of pass profile vectors $\{y_k\}_k$.

The dynamics of repetitive processes evolve over a restricted quadrant of the positive quadrant and have a number of practical applications, for example they arise naturally in the modelling of long-wall coal cutting (for background on this and other physical examples see the references given in [5]). A number of so-called algorithmic examples also exist where adopting a repetitive process setting for analysis has clear advantages over alternative approaches to systems related analysis. These include iterative learning control (ILC) schemes, e.g. [6] and iterative solution algorithms for dynamic nonlinear optimal control problems based on the maximum principle, e.g. [7]. In the case of iterative learning control for the linear dynamics case, the stability theory for differential (and discrete) linear repetitive processes is one method which can be used to undertake a stability/convergence analysis of a powerful class of such algorithms and thereby produce vital design information concerning the trade-offs required between convergence and transient performance (see e.g. [8]).

A considerable degree of progress has been made in the development of a control and systems theory for these repetitive processes together with computational design algorithms. For progress here again see [5, Chapters 8–10] and note also that ILC laws developed using this setting have been experimentally verified.

In this paper the novel contributions are (i) control law design algorithms for linear repetitive processes where, in effect, the repetitions propagate a 2D plane of information and hence what we finish up considering is a 3D system and (ii) the results of applying this repetitive process model and resulting control law designs to distributed parameter systems focusing on fourth-order wavefront equations arising, for example, from discretization of the partial differential equation (PDE) which describes lateral deflection of a thin flexible plate.

2. Distributed parameter dynamics as repetitive processes/nD systems

The discrete linear repetitive processes considered in this paper have the following state-space model:

$$x_{k+1}(l,m) = \sum_{i=-\varepsilon}^{\varepsilon} \sum_{j=-\varepsilon}^{\varepsilon} (A^{i,j} x_k(l+i, m+j) + B^{i,j} u_k(l+i, m+j)), \tag{1}$$

where on pass k, $x_k(l,m) \in \mathbb{R}^n$ is the state vector, $u_k(l,m) \in \mathbb{R}^q$ is the control input vector, and $\varepsilon > 0$ is a positive integer. The boundary conditions are

$$\begin{aligned}
&x_k(l,m) = 0, \quad -\varepsilon \leqslant l < 0,\ 0 \leqslant m \leqslant \beta,\ k \geqslant 0,\\
&x_k(l,m) = 0, \quad -\varepsilon \leqslant m < 0,\ 0 \leqslant l \leqslant \alpha,\ k \geqslant 0,\\
&x_0(l,m) \doteq d_0(l,m), \quad 0 \leqslant l \leqslant \alpha,\ 0 \leqslant m \leqslant \beta,\\
&x_k(\alpha - i, m) \doteq d_k(i,m), \quad 0 \leqslant m \leqslant \beta,\ 0 \leqslant i < \varepsilon,\ k \geqslant 0,\\
&x_k(l, \beta - j) \doteq d_k(l,j), \quad 0 \leqslant l \leqslant \alpha,\ 0 \leqslant j < \varepsilon,\ k \geqslant 0.
\end{aligned} \tag{2}$$

Here the process dynamics are defined over a finite fixed rectangle, i.e. $0 \leqslant l \leqslant \alpha - \varepsilon, 0 \leqslant m \leqslant \beta - \varepsilon$ but at any point on pass $k+1$ it is only the points in the so-called mask $-\varepsilon \leqslant l \leqslant \varepsilon, -\varepsilon \leqslant m \leqslant \varepsilon$ on the previous pass which contribute to the pass profile. The updating structure for the case when $\varepsilon = 1$ is illustrated in Fig. 1. (Note also that the results in this paper are easily generated to the case when the mask is a rectangle.)

In these processes therefore it is a plane, or rectangle, of information which is propagated in the pass-to-pass direction. Note also that these processes share many joint features with the so-called spatially interconnected systems, which have already found numerous important physical applications, see, for example, [9] and references therein. This arises from the fact that some of the state-space models in this latter area can be rewritten as a discrete linear repetitive process state-space model (or its differential equivalent). Next we show how such a model structure arises in the modelling for control of mechanical systems (and also in electrical and electro-mechanical systems, etc).

Consider a thin flexible plate of the form shown in Fig. 2 subject to a transverse external force. Then the resulting deformation dynamics are modelled using a PDE of the following form first obtained by Lagrange in 1811 (see, for example, [10] for full details):

$$\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4} + \frac{\rho}{D}\frac{\partial^2 w(x,y,t)}{\partial t^2} = \frac{q(x,y,t)}{D}, \tag{3}$$

where w is the lateral deflection in the z direction (m), ρ is the mass density per unit area (kg/m^2), q is the transverse external force, with dimension of force per unit area (N/m^2), $(\partial^2 w/\partial t^2)$ is the acceleration in the z direction (m/s^2), $D = Eh^3/(12(1-v^2))$, v is Poisson's ratio, h is thickness of the plate (m), E is Young's modulus (N/m^2).

If control action is to be applied, then this will be implemented digitally and hence Eq. (3) must be discretized with respect to time. Moreover, if an array of actuators and zonal type wavefront sensors are to be used, discretization in the spatial variables is also required.

Finite difference (FD) methods are a well established numerical tool for solving PDEs (see, for instance, [11]). The basic principle of these methods is to cover the region where a solution is sought by a regular grid and to replace derivatives by differences using only values at the nodal points. There are many types of grids which can be used, e.g. rectangular, hexagonal, triangular or polar. Of these, the rectangular one is very appealing because of very simple difference formulas which result. However, triangular or hexagonal

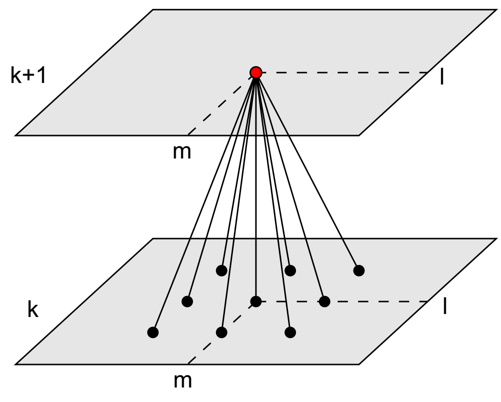

Fig. 1. Updating structure.

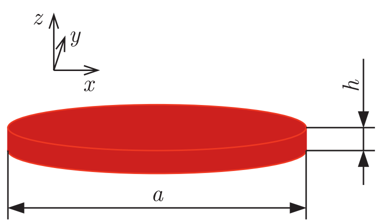

Fig. 2. Thin circular plate.

grids are better fitted to the circular aperture and here we will consider a circular thin flexible plate and a triangular grid and derive corresponding difference formulas to obtain a recurrence equation approximating the PDE (3).

Consider a triangular grid employed according to Fig. 3 and denote the number of nodal points on the plate bisector by n. Further, let Δx and Δy denote the distance between the nodes in the x and y directions, respectively, and let the sampling (time) period be Δt.

For ease of notation we now use the subscripts l and m to denote the sample number of the spatial variables x and y respectively, and the subscript k to denote the sample number in the time variable t. In the time domain, the central difference approximation is used and the corresponding formula is

$$\left(\frac{\partial^2 w}{\partial t^2}\right)_{l,m,k} = \frac{1}{\Delta t^2}(w_{l,m,k+1} - 2w_{l,m,k} + w_{l,m,k-1}). \tag{4}$$

In the space domain, we can use only values at the nodal points and the choice of these is not unique. Here we use the following formulas:

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{l,m,k} = \frac{1}{\Delta x^4}(6w_{l,m,k} - 2w_{l+1,m+1,k} - 2w_{l+1,m-1,k} - 2w_{l-1,m+1,k} - 2w_{l-1,m+1,k} + w_{l-2,m,k} + w_{l+2,m,k}), \tag{5}$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{l,m,k} = \frac{1}{\Delta y^4}(6w_{l,m,k} - 2w_{l+1,m+1,k} - 2w_{l+1,m-1,k} - 2w_{l-1,m+1,k} - 2w_{l-1,m+1,k} + w_{l,m-2,k} + w_{l,m+2,k}), \tag{6}$$

$$\left(\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)_{l,m,k} = \frac{1}{\Delta x^2 \Delta y^2}(4w_{l,m,k} - w_{l+1,m+1,k} - w_{l+1,m-1,k} - w_{l-1,m+1,k} - w_{l-1,m+1,k}). \tag{7}$$

Substitution of Eqs. (4) through (7) into Eq. (3) gives the recurrence equation

$$\begin{aligned} w_{l,m,k+1} = & -\frac{D\Delta t^2}{\rho}[Pw_{l,m,k} + Q(w_{l-1,m-1,k} + w_{l-1,m+1,k} + w_{l+1,m-1,k} + w_{l+1,m+1,k}) + R(w_{l-2,m,k} + w_{l+2,m,k}) \\ & + S(w_{l,m-2,k} + w_{l,m+2,k})] + 2w_{l,m,k} - w_{l,m,k-1} + \frac{\Delta t^2}{\rho}q_{l,m,k}, \end{aligned} \tag{8}$$

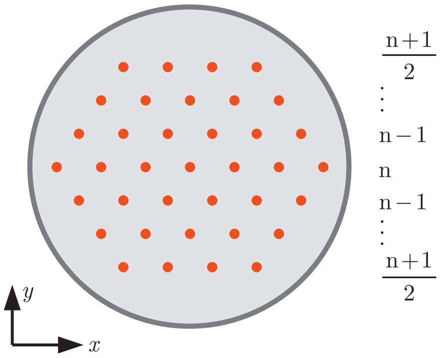

Fig. 3. An example of the triangular grid for $n = 7$ (the number of nodal points in a row is given on the right-hand side).

where

$$
\begin{aligned}
P &= \frac{6}{\Delta x^4} + \frac{8}{\Delta x^2 \Delta y^2} + \frac{6}{\Delta y^4}, \\
Q &= -\frac{2}{\Delta x^4} - \frac{2}{\Delta x^2 \Delta y^2} - \frac{2}{\Delta y^4}, \\
R &= \frac{1}{\Delta x^4}, \quad S = \frac{1}{\Delta y^4}.
\end{aligned}
$$

In the most practical situations, the triangular grid will consist of equilateral triangles, i.e.

$$\Delta y = \sqrt{3}\Delta x \tag{9}$$

and in this case the coefficients P, Q, R, S simplify to

$$P = \frac{28}{3} \cdot \frac{1}{\Delta x^4}, \quad Q = -\frac{26}{9} \cdot \frac{1}{\Delta x^4}, \quad R = \frac{1}{\Delta x^4}, \quad S = \frac{1}{9} \cdot \frac{1}{\Delta x^4}.$$

Before proceeding, it is essential to verify the model just obtained and, in particular, if it is an acceptably precise approximation to the original model described by the PDE. This is established by means of a stability analysis of the iterative FD scheme, the objective being to determine whether the iterative scheme given by Eq. (8) converges to a solution. In particular, we determine a relationship between Δt and Δx which guarantees convergence.

With zero external force applied, Eq. (8) becomes

$$
\begin{aligned}
&-\frac{D\Delta t^2}{\rho}[P w_{l,m,k} + Q(w_{l-1,m-1,k} + w_{l-1,m+1,k} + w_{l+1,m-1,k} + w_{l+1,m+1,k}) \\
&\quad + R(w_{l-2,m,k} + w_{l+2,m,k}) + S(w_{l,m-2,k} + w_{l,m+2,k})] \\
&\qquad = w_{l,m,k+1} - 2w_{l,m,k} + w_{l,m,k-1}
\end{aligned} \tag{10}
$$

and we now apply von Neumann stability analysis (a standard technique in this general area). Replacing $w_{l,m,k}$ in Eq. (10) by $g^k e^{jl\theta_1} e^{jm\theta_2}$ gives

$$
\begin{aligned}
&-\frac{D\Delta t^2}{\rho}[P g^k e^{jl\theta_1} e^{jm\theta_2} + Q g^k (e^{j(l-1)\theta_1} e^{j(m-1)\theta_2} + e^{j(l+1)\theta_1} e^{j(m-1)\theta_2} + e^{j(l-1)\theta_1} e^{j(m+1)\theta_2} + e^{j(l+1)\theta_1} e^{j(m+1)\theta_2}) \\
&\quad + R g^k (e^{j(l-2)\theta_1} e^{jm\theta_2} + e^{j(l-2)\theta_1} e^{jm\theta_2}) + S g^k (e^{jl\theta_1} e^{j(m-2)\theta_2} + e^{jl\theta_1} e^{j(m+2)\theta_2})] \\
&\qquad = g^{k+1} e^{jl\theta_1} e^{jm\theta_2} - 2g^k e^{j(l-1)\theta_1} e^{j(m-1)\theta_2} + g^{k-1} e^{j(l-1)\theta_1} e^{j(m-1)\theta_2},
\end{aligned}
$$

where θ_1 and θ_2 are the spatial frequencies in the x and y directions, respectively, and $j = \sqrt{-1}$. The parameter g is called the amplification factor and the scheme is stable if and only if $|g| \leqslant 1$, see [11] for details. Using Euler's formula and some routine simplification analysis now gives the amplification factor as

$$g_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{11}$$

where

$$
\begin{aligned}
a &= 1, \\
b &= \frac{2D\cos(2\theta_1)}{\Delta x^4 \rho} - \frac{2}{\Delta t^2} + \frac{2D\cos(2\theta_2)}{9\Delta x^4 \rho} + \frac{56D\cos(\theta_1 + \theta_2)}{3\Delta x^4 \rho} - \frac{52D\cos(\theta_1 - \theta_2)}{9\Delta x^4 \rho}, \\
c &= \frac{1}{\Delta t^2}.
\end{aligned}
$$

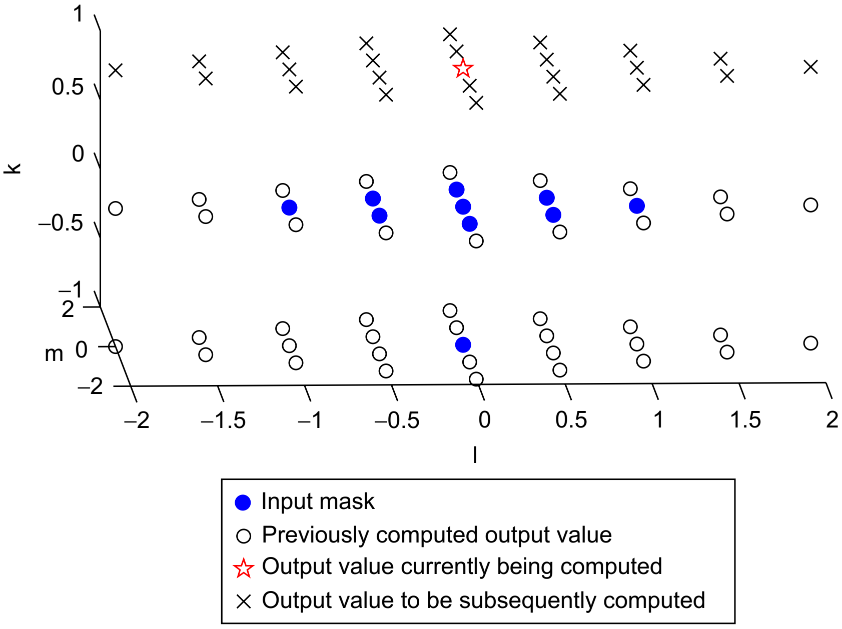

Fig. 4. Computation mask associated with partial recurrence equation (8).

The absolute value of Eq. (11) reaches its maximum for $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$ and the corresponding Δt is

$$\Delta t \leqslant \frac{3\sqrt{3\rho}\Delta x^2}{\sqrt{136D - 9\rho\Delta x^4}}. \tag{12}$$

Eq. (8) is clearly a special case of the repetitive process model (1) where $\varepsilon = 2$, and, for example, $w_{l,m,k} \to x_k(l,m)$ and $q_{l,m,k} \to u_k(l,m)$. In particular, this repetitive process model approximately describes the transverse vibrations of the plate and Fig. 4 shows the associated computation mask, i.e. the evolution of the updating structure in the repetitive process model.

3. Stability analysis and control law design

Stability analysis for the repetitive processes considered here is based on so-called quadratic stability. In particular, given matrices

$$V^{i,j} > 0, \quad \forall i = -\varepsilon, \ldots, \varepsilon, \; j = -\varepsilon, \ldots, \varepsilon \tag{13}$$

(where we denote a symmetric positive definite matrix, say X, by $X > 0$) introduce the following so-called "local" Lyapunov function

$$V_k(l,m) \doteq \sum_{i=-\varepsilon}^{\varepsilon} \sum_{j=-\varepsilon}^{\varepsilon} x_k^T(l+i, m+j) V^{i,j} x_k(l+i, m+j). \tag{14}$$

This function is the local energy for the considered mask (i.e. $-\varepsilon \leqslant l \leqslant \varepsilon, -\varepsilon \leqslant m \leqslant \varepsilon$). The so-called total Lyapunov function is

$$\mathscr{V}_k \doteq \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} x_k^T(i,j) V x_k(i,j), \tag{15}$$

where V is defined by Eq. (18) below.

Motivated by physical arguments that the total energy at the pass (finite for all of them) should decrease from pass to pass we introduce the following definition of quadratic stability.

Definition 1. A discrete linear repetitive process described by Eqs. (1) and (2) is said to be quadratically stable provided $\mathscr{V}_0 < \infty$ and there exist matrices $V^{i,j} > 0, i = -\varepsilon, \dots, \varepsilon, j = -\varepsilon, \dots, \varepsilon$ such that

$$\mathscr{V}_{k+1} - \mathscr{V}_k < 0 \tag{16}$$

for all $x_k(l,m)(\neq 0) \in \mathbb{R}^n$.

To develop a computationally efficient test for this property, the associated increment for the local Lyapunov function is defined as

$$\Delta V_k(l,m) \doteq x_{k+1}^T(l,m) V x_{k+1}(l,m) - \sum_{i=-\varepsilon}^{\varepsilon} \sum_{j=-\varepsilon}^{\varepsilon} x_k^T(l+i,m+j) V^{i,j} x_k(l+i,m+j), \tag{17}$$

where

$$V \doteq \sum_{i=-\varepsilon}^{\varepsilon} \sum_{j=-\varepsilon}^{\varepsilon} V^{i,j}. \tag{18}$$

Now we have the following first major result.

Theorem 1. *A discrete linear repetitive process described by Eqs.* (1) *and* (2) *is quadratically stable if there exist matrices* $V^{i,j} > 0, i = -\varepsilon, \dots, \varepsilon, j = -\varepsilon, \dots, \varepsilon$ *such that*

$$\Delta V_k(l,m) < 0, \quad \forall 0 \leqslant l \leqslant \alpha, \ 0 \leqslant m \leqslant \beta, \ k \geqslant 0 \tag{19}$$

for all $x_k(l,m) \in \mathbb{R}^n$.

Proof. It is straightforward to check that summing the increments $\Delta V_k(l,m)$ over all points $l,m \quad 0 \leqslant l \leqslant \alpha, 0 \leqslant m \leqslant \beta$, for given pass k and taking into account the boundary conditions yields the total Lyapunov function increment $\mathscr{V}_{k+1} - \mathscr{V}_k < 0$. $\square$

This result can also be represented in the form of a linear matrix inequality (LMI), which provides a computational test for this property.

Theorem 2. *A discrete linear repetitive process described by Eqs.* (1) *and* (2) *is quadratically stable if there exist* $V^{i,j} > 0$, $\forall i \in \{-\varepsilon, \dots, 0, \dots, \varepsilon\}, \forall j \in \{-\varepsilon, \dots, 0, \dots, \varepsilon\}$ *such that the following LMI holds*:

$$\mathbf{A}^T \mathbf{V} \mathbf{A} - \mathbf{V} < 0, \tag{20}$$

where

$$\mathbf{V} \doteq \bigoplus_{i=-\varepsilon}^{\varepsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} V^{i,j} \tag{21}$$

and $\bigoplus$ *denotes the direct sum of matrices, i.e. for two matrices say* X_1 *and* X_2

$$X_1 \bigoplus X_2 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},$$

$$\mathbf{A} \doteq \begin{bmatrix} A^{-\varepsilon,-\varepsilon} & \cdots & A^{-\varepsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ A^{\varepsilon,-\varepsilon} & \cdots & A^{\varepsilon,\varepsilon} \end{bmatrix}. \tag{22}$$

Proof. It is easy to check that the condition of Definition 1 holds if Eq. (20) holds. $\square$

Suppose that a control law of the following form is applied to a process described by Eq. (1):

$$\begin{bmatrix} u_k^{-\varepsilon,-\varepsilon}(l,m) \\ \vdots \\ u_k^{0,0}(l,m) \\ \vdots \\ u_k^{\varepsilon,\varepsilon}(l,m) \end{bmatrix} = \mathbf{K} \begin{bmatrix} x_k(l-\varepsilon,m-\varepsilon) \\ \vdots \\ x_k(l,m) \\ \vdots \\ x_k(l+\varepsilon,m+\varepsilon) \end{bmatrix}, \tag{23}$$

where

$$\mathbf{K} \doteq \bigoplus_{i=-\varepsilon}^{\varepsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} K^{i,j}. \tag{24}$$

Then interpreting Eq. (20) in terms of the resulting state-space model of the controlled process gives the following sufficient condition for quadratic stability:

$$(\mathbf{A}+\mathbf{BK})^T\mathbf{V}(\mathbf{A}+\mathbf{BK}) - \mathbf{V} < 0, \tag{25}$$

where the matrix $\mathbf{B}$ is given by

$$\mathbf{B} \doteq \begin{bmatrix} B^{-\varepsilon,-\varepsilon} & \cdots & B^{-\varepsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ B^{\varepsilon,-\varepsilon} & \cdots & B^{\varepsilon,\varepsilon} \end{bmatrix} \tag{26}$$

and we have the following result.

Theorem 3. *Suppose that a control law of the form Eq.* (23) *is applied to a discrete linear repetitive process described by Eqs.* (1) *and* (2). *Then the resulting controlled process is quadratically stable if there exists a block diagonal matrix* $\mathbf{X}$ *which contains symmetric and positive definite matrices* $X^{i,j} > 0$, *for all* $i \in \{-\varepsilon, \ldots, 0, \ldots, \varepsilon\}$, *for all* $j \in \{-\varepsilon, \ldots, 0, \ldots, \varepsilon\}$

$$\mathbf{X} \doteq \bigoplus_{i=-\varepsilon}^{\varepsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} X^{i,j} \tag{27}$$

and

$$\mathbf{N} \doteq \bigoplus_{i=-\varepsilon}^{\varepsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} N^{i,j} \tag{28}$$

such that

$$\begin{bmatrix} -\mathbf{X} & \mathbf{XA}^T + \mathbf{N}^T\mathbf{B}^T \\ \mathbf{AX}+\mathbf{BN} & -\mathbf{X} \end{bmatrix} < 0. \tag{29}$$

If this condition holds, a stabilizing $\mathbf{K}$ *in the control law* (23) *is given by*

$$\mathbf{K} = \mathbf{NX}^{-1}. \tag{30}$$

Proof. Follows immediately as a result of (i) an obvious application of the Schur's complement formula [13] to Eq. (25), (ii) the application of appropriate congruence transformations to the result of (i), and (iii) substitution from Eq. (23). □

3.1. An alternative approach to control law design

Better computational results can be obtained based on [14] and first adopted for repetitive processes in [15].

Theorem 4. *The condition of Theorem* 2 *is equivalent to the existence of matrices* $\mathbf{V}>0$ (*defined in Theorem* 2) *and* $\mathbf{G}$ *such that*

$$\begin{bmatrix} -\mathbf{V} & \mathbf{AG} \\ \mathbf{G}^T\mathbf{A}^T & -\mathbf{G}-\mathbf{G}^T+\mathbf{V} \end{bmatrix} < 0, \tag{31}$$

where

$$\mathbf{G} \doteq \bigoplus_{i=-\varepsilon}^{\varepsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} G^{i,j}. \tag{32}$$

Proof. Assume that Eq. (31) is feasible. Then

$$-\mathbf{G}-\mathbf{G}^T+\mathbf{V}<0$$

and, since $\mathbf{G}$ is full rank and $\mathbf{V}>0$, we have that

$$(\mathbf{V}-\mathbf{G})^T\mathbf{V}^{-1}(\mathbf{V}-\mathbf{G}) \geqslant 0$$

or, equivalently,

$$-\mathbf{G}^T\mathbf{V}^{-1}\mathbf{G} \leqslant -\mathbf{G}-\mathbf{G}^T+\mathbf{V}.$$

Hence the following LMI is a sufficient condition for Eq. (31) to hold:

$$\begin{bmatrix} -\mathbf{V} & \mathbf{AG} \\ \mathbf{G}^T\mathbf{A}^T & -\mathbf{G}^T\mathbf{V}^{-1}\mathbf{G} \end{bmatrix} < 0.$$

Left and right multiplication of this last result by

$$\begin{bmatrix} \mathbf{V}^{-1} & 0 \\ 0 & \mathbf{G}^{-T} \end{bmatrix}$$

and its transpose, respectively, and then setting $W=\mathbf{V}^{-1}>0$ yields

$$\begin{bmatrix} -W & W\mathbf{A} \\ \mathbf{A}^T W & -W \end{bmatrix} < 0 \tag{33}$$

which is equivalent to Eq. (20), and hence the sufficiency part of the proof is complete.

To prove necessity, assume that Eq. (16) is satisfied. Then

$$W-\mathbf{A}^T W\mathbf{A}>0$$

which can be rewritten as

$$\mathbf{V}-\mathbf{AVA}^T>0$$

on setting $\mathbf{V}=W^{-1}$. Next, introduce $\mathbf{G}=\mathbf{V}+gI$, where g is a positive scalar. Then, there exists a sufficiently small g such that

$$g^{-2}(\mathbf{V}+2gI)>\mathbf{A}^T T^{-1}\mathbf{A}$$

which is equivalent, by the Schur's complement formula, to

$$\begin{bmatrix} T & -g\mathbf{A} \\ -g\mathbf{A}^T & \mathbf{V}+2gI \end{bmatrix} > 0$$

or

$$\begin{bmatrix} \mathbf{V}-\mathbf{AVA}^T & \mathbf{AV}-\mathbf{AG} \\ \mathbf{VA}^T-\mathbf{GA}^T & \mathbf{G}+\mathbf{G}^T-\mathbf{V} \end{bmatrix} > 0.$$

This last LMI can be written as

$$\begin{bmatrix} I & -\mathbf{A} \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{V} & \mathbf{AG} \\ \mathbf{G}^T\mathbf{A}^T & \mathbf{G}+\mathbf{G}^T-\mathbf{V} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\mathbf{A}^T & I \end{bmatrix}>0$$

and hence necessity is established and the proof is complete. □

Suppose now that we have a control law of Eq. (23) and it is applied to a process described by Eqs. (1) and (2). Then we have the following result.

Theorem 5. *Suppose that a control law of the form* (23) *is applied to a discrete linear repetitive process described by Eqs.* (1) *and* (2). *Then the resulting controlled process is quadratically stable if there exists a block diagonal matrix* $\mathbf{V}>0$ (*defined in Theorem* 2), *a matrix* $\mathbf{G}$ (*defined in Theorem* 4) *and a matrix* $\mathbf{N}$ (*defined in Theorem* 3) *such that*

$$\begin{bmatrix} -\mathbf{V} & \mathbf{AG}+\mathbf{BN} \\ (\mathbf{AG}+\mathbf{BN})^T & -\mathbf{G}-\mathbf{G}^T+\mathbf{V} \end{bmatrix}<0. \tag{34}$$

If this condition holds, a stabilizing $\mathbf{K}$ *in the control law* (23) *is given by*

$$\mathbf{K}=\mathbf{NG}^{-1}. \tag{35}$$

Proof. This follows immediately from interpreting Theorem 4 in terms of the controlled process and then setting $\mathbf{KG}=\mathbf{N}$. □

4. A numerical example

Consider the case when the plate parameters are given in Table 1.

Suppose also that the initial plate deflection is zero, i.e. the forces and moments acting on the plate due to its weight are neglected and hence the initial condition is

$$w_{l,m,k}|_{t=0}=0.$$

Suppose also that the edge of the plate is clamped. Then the plate deflection on the edge is always equal to zero as is its derivative. The boundary conditions are

$$w(x,y,t)|_{x,y\in\partial D}=0, \quad \left.\frac{\partial w(x,y,t)}{\partial x}\right|_{x,y\in\partial D}=0,$$

$$\left.\frac{\partial w(x,y,t)}{\partial y}\right|_{x,y\in\partial D}=0,$$

where $\partial\mathscr{D}$ denotes the boundary of the region where we wish to find a solution. At every boundary point, the following conditions must hold:

$$w_{l,m,k}=0,$$

$$w_{l-1,m-1,k}+w_{l-1,m+1,k}-w_{l+1,m-1,k}-w_{l+1,m+1,k}=0.$$

Table 1
Plate parameters

Parameter	Value
Diameter (a)	1 m
Thickness (h)	0.003 m
Mass density per unit area (ρ)	2700 kg/m^2
Young's modulus (E)	7.11×10^{-11} m^2
Poisson ratio (v)	0.3

Under these conditions discretization by triangular grid with $n = 25$ is an appropriate compromise between obtaining a good approximation without excessive computing time and storage requirements. The node distances in the x and y directions are

$$\Delta x = \frac{1}{2}\frac{a}{n+1}, \quad \Delta y = \frac{\sqrt{3}}{2}\frac{a}{n+1},$$

respectively, and hence $\Delta x = 0.0192\,\mathrm{m}$ and $\Delta y = 0.0333\,\mathrm{m}$. The sampling period was chosen as $\Delta t = 1 \times 10^{-4}\,\mathrm{s}$ which satisfies Eq. (12), since

$$0 < \Delta t \leqslant 5.8608 \times 10^{-4}.$$

To obtain the response to nonzero initial conditions, the eigenfunction corresponding to the smallest frequency was computed. This has the form

$$v_k = \frac{1}{A|\mathrm{J}_0(\mu_k)|\mathrm{I}_0(\mu_k)}\left[\mathrm{J}_0\left(\mu_k\frac{r}{A}\right) - \frac{\mathrm{J}_0(\mu_k)}{\mathrm{I}_0(\mu_k)}\mathrm{I}_0\left(\mu_k\frac{r}{A}\right)\right], \quad k \in \mathbb{N}, \ 0 < r < A, \ A = \frac{a}{2},$$

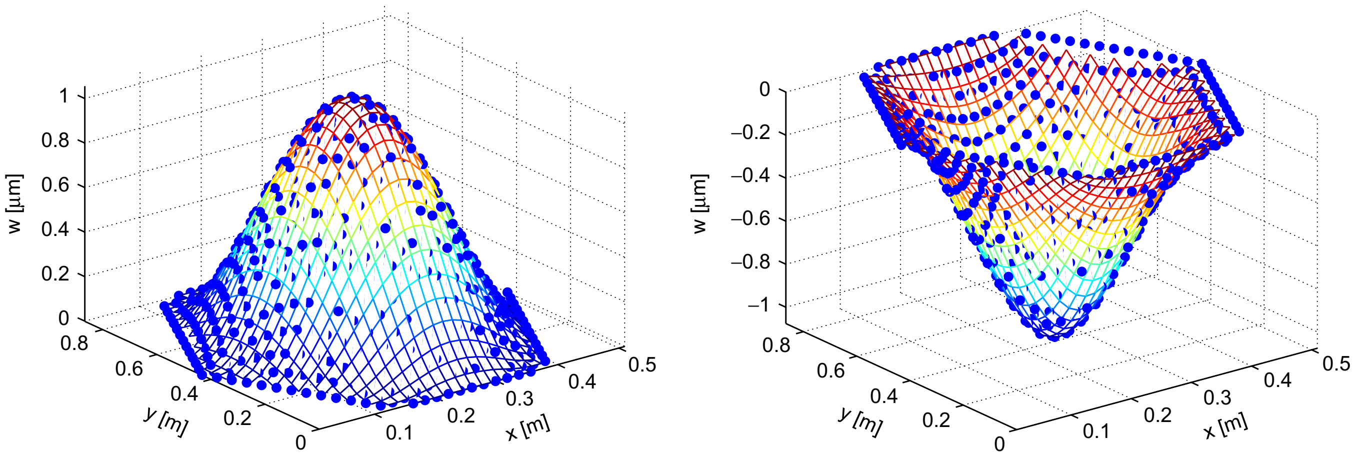

Fig. 5. Plate deflection at $t = 0$ and 0.052 s.

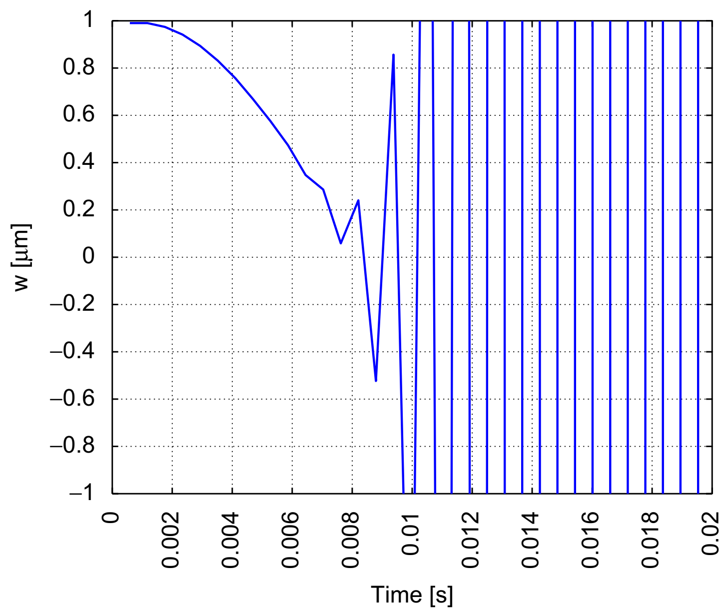

Fig. 6. Plate deflection at the middle point, condition (12) does not hold.

where J_0 and I_0 are Bessel and modified Bessel functions of index zero, respectively, see, for example, [12] for details of this standard approach. The eigenfunction corresponding to the smallest frequency can be written as

$$v_1 = \frac{1}{A|J_0(\mu_1)|I_0(\mu_1)}\left[J_0\left(\mu_1\frac{r}{A}\right) - \frac{J_0(\mu_1)}{I_0(\mu_1)}I_0\left(\mu_1\frac{r}{A}\right)\right],$$

where $\mu_1 \doteq 3.190$.

Fig. 5 shows the scaled deflection of the plate at the beginning (left plot) of the simulation and after 0.052 s (right plot). Fig. 6 shows the scaled deflection of the plate when the condition of Eq. (12) is not satisfied.

Consider now the application of Theorem 2 to the numerical example specified above. Then the corresponding LMI does not have a solution and hence we proceed to consider the design of a stabilizing control law using Theorem 5. In order to do this we must use a mapping from the triangular grid used to approximate the process dynamics to the linear ordering used in Theorems 2 and 5. It is hence convenient to define the function $\varphi(\omega)\mapsto\{i,j\}$ which maps indices as follows:

$$\begin{aligned}
&\varphi(1)\mapsto\{0,-2\}, \quad \varphi(2)\mapsto\{-1,-1\},\\
&\varphi(3)\mapsto\{1,-1\}, \quad \varphi(4)\mapsto\{-2,0\},\\
&\varphi(5)\mapsto\{0,0\}, \quad \varphi(6)\mapsto\{2,0\},\\
&\varphi(7)\mapsto\{-1,1\}, \quad \varphi(8)\mapsto\{1,1\},\\
&\varphi(9)\mapsto\{0,2\}
\end{aligned} \tag{36}$$

and additionally

$$\begin{aligned}
&\varphi(\omega,1)\mapsto i,\\
&\varphi(\omega,2)\mapsto j.
\end{aligned} \tag{37}$$

For example, $\varphi(7,1)\mapsto -1$ and $\varphi(7,2)\mapsto 1$. Then we have

$$\mathbf{A} \doteq \begin{bmatrix} A^{\varphi(1)} & \cdots & A^{\varphi(9)} \\ \vdots & \ddots & \vdots \\ A^{\varphi(1)} & \cdots & A^{\varphi(9)} \end{bmatrix}, \tag{38}$$

$$\mathbf{B} \doteq \begin{bmatrix} B^{\varphi(1)} & \cdots & B^{\varphi(9)} \\ \vdots & \ddots & \vdots \\ B^{\varphi(1)} & \cdots & B^{\varphi(9)} \end{bmatrix}, \tag{39}$$

where the 2×2 matrices $A^{\varphi(\omega)}$, $B^{\varphi(\omega)}$ and $\omega = 1, 2, \ldots, 9$ are constructed from the appropriate coefficients of the underlying discrete equation as

$$A^{\varphi(1)} = A^{\varphi(9)} = \begin{pmatrix} -\frac{D\Delta t^2}{\rho}S & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -6.4\times 10^{-4} & 0 \\ 0 & 0 \end{pmatrix},$$

$$A^{\varphi(2)} = A^{\varphi(3)} = A^{\varphi(7)} = A^{\varphi(8)} = \begin{pmatrix} -\frac{D\Delta t^2}{\rho}Q & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.013 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A^{\varphi(5)} = \begin{pmatrix} -\frac{D\Delta t^2}{\rho}P + 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1.9 & -1 \\ 1 & 0 \end{pmatrix},$$

$$A^{\varphi(4)} = A^{\varphi(6)} = \begin{pmatrix} -\dfrac{D\Delta t^2}{\rho}R & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -5.8 \times 10^{-3} & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^{\varphi(1)} = B^{\varphi(2)} = B^{\varphi(3)} = B^{\varphi(4)} = B^{\varphi(5)} = B^{\varphi(6)} = B^{\varphi(7)} = B^{\varphi(8)} = B^{\varphi(9)}$$
$$= \begin{pmatrix} \dfrac{\Delta t^2}{\rho} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3.7 \times 10^{-12} & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose now that a control law of the following form is applied:

$$\begin{bmatrix} u_k^{\varphi(1)}(l,m) \\ \vdots \\ u_k^{\varphi(9)}(l,m) \end{bmatrix} = \mathbf{K} \begin{bmatrix} x_k(l+\varphi(1,1), m+\varphi(1,2)) \\ \vdots \\ x_k(l+\varphi(9,1), m+\varphi(9,2)) \end{bmatrix}, \tag{40}$$

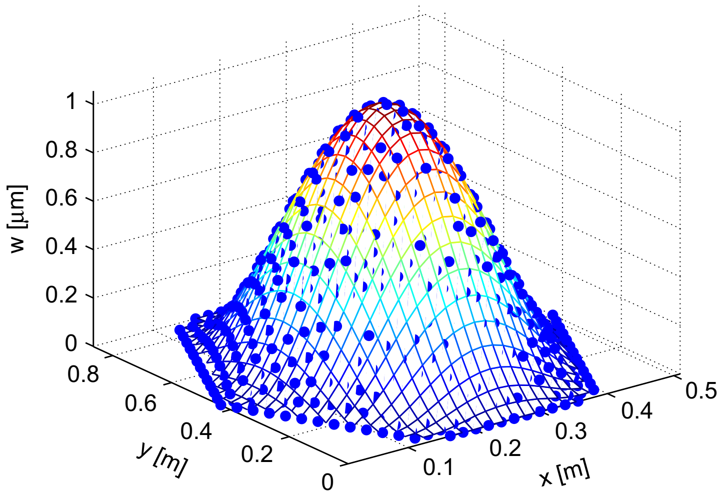

Fig. 7. Initial conditions.

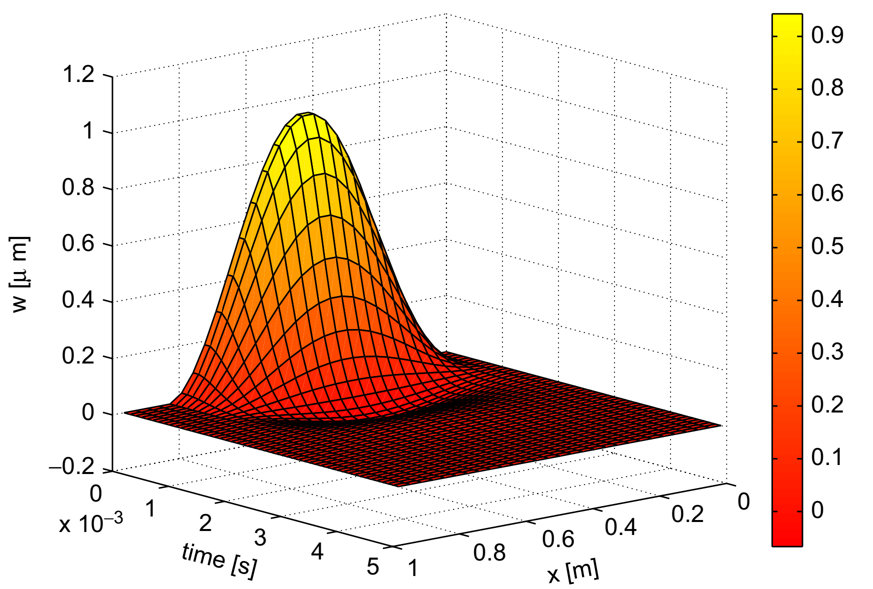

Fig. 8. Deflection at nodes on the middle diagonal—controlled system with control law (40).

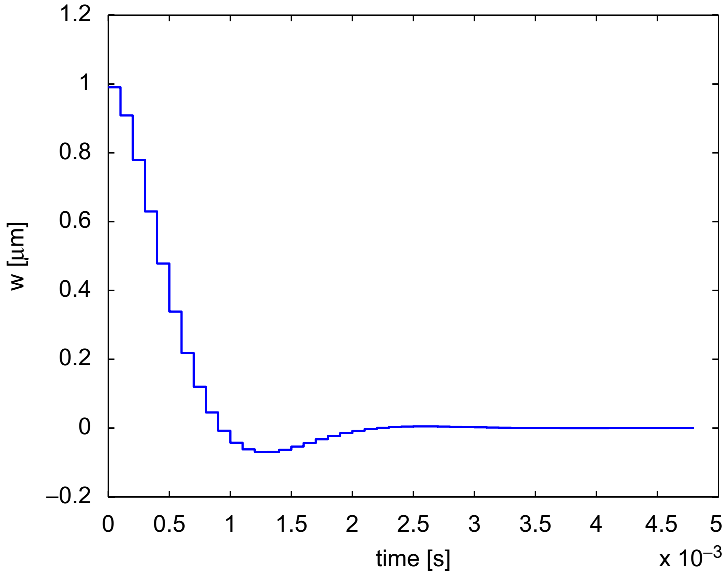

Fig. 9. Deflection at a node in the middle of the plate—controlled process with control law (40).

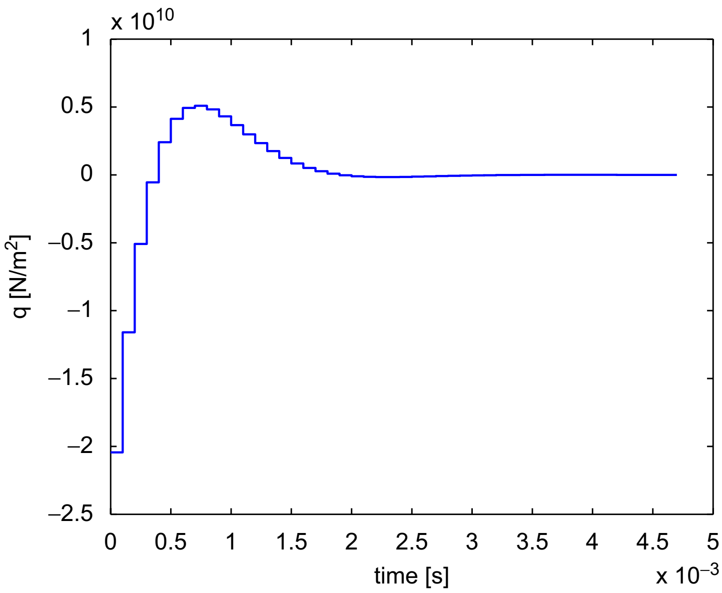

Fig. 10. Control signal at node in the middle of the plate—controlled process with control law (40).

where

$$\mathbf{K} \doteq \bigoplus_{\omega=1}^{9} K^{\varphi(\omega)}. \tag{41}$$

Then the LMI of Theorem 5 has a solution and stabilizing control law matrices are given by

$$K^{\varphi(1)} = \begin{pmatrix} -1.2 \times 10^{11} & 0.97 \times 10^{11} \\ 0 & 0 \end{pmatrix},$$

$$K^{\varphi(\omega)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \omega = 2, 3, \ldots, 9.$$

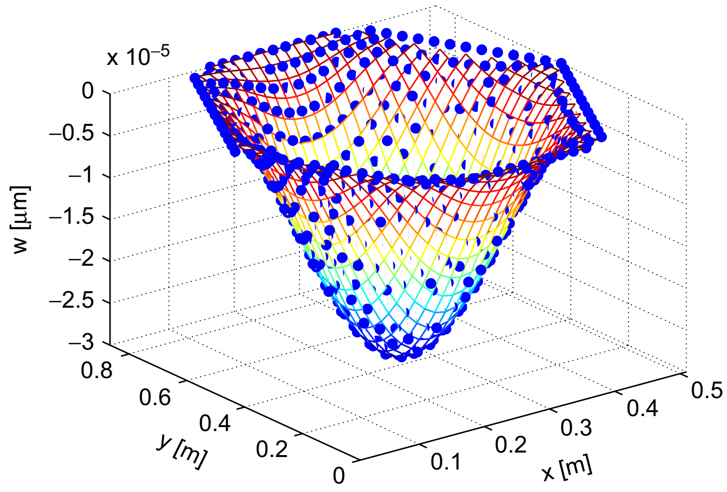

Fig. 11. Deflection of the plate at time 5 ms—controlled process with control law (40).

Fig. 7 shows initial conditions used for the simulation study reported here for this numerical example. Figs. 8, 9 and 10 show response of the controlled response at nodes on the middle diagonal, the deflection at a node in the middle of the plate, and the control signal at the same node in the middle of the plate, respectively. Fig. 11 shows deflection of the complete plate after 5 ms. These confirm that a stabilizing control law has been produced and since it is a regulator problem, the initial deflection is eventually returned to rest.

5. Conclusions

This paper has produced the first substantial results on a new model for repetitive processes where it is a plane of information which is propagated in the pass-to-pass direction. This makes the system three dimensional (3D) and motivation for considering such a model has been given by showing how it can arise in the discretization of the dynamics of distributed parameter systems. This is in the form of a fourth-order partial differential equation which arises in the modelling of the transverse vibrations of a thin plate.

Quadratic stability for this new repetitive process model has been defined in energy terms and it has been shown that the resulting condition can be expressed in terms of an LMI. Moreover, this also provides a basis on which to specify and design control laws for distributed parameter systems with, in particular, immediate recourse to well documented and powerful computational tools in the form of LMIs. The analysis here is based on sufficient but not necessary stability conditions and hence a degree of conservativeness could be present but experience in other repetitive process theory strongly suggests that this is often not very severe.

The results in this paper are the first on this form of repetitive process dynamics and much remains to be done both in terms of theory and also potential applications. This is especially true given the emphasis now on distributed control for application to, for example, adaptive optics systems (see, for example, [16] for background) where [17] contains some results from analysis in an nD systems setting (this is based on polynomial methods and is hence limited in terms of cases to which design can be completed). Other potential application areas for a repetitive process based approach to the control of distributed parameter systems include scene based iterative learning control [18] and also diffusion control in irrigation applications [19]. Also, via the connection to iterative learning control, the repetitive process setting can be used in repetitive control (for possibly relevant work see [20]).

Progress here will only be feasible after much further research is completed. Obvious areas for this include (i) the discretization methods possible since FE methods may often not be appropriate or even applicable and the question then to be answered is can we again get to a repetitive process model approximation to the dynamics which is suitable and realistic basis for control law design, (ii) the use of model validation tools beyond the classical von Neumann approach used here, (iii) exactly what classes of partial differential equations can be treated in this way, (iv) robust control design since we will always be using an approximate model for design and initial control law evaluation, and (v) comparison (where applicable) with alternative approaches, such as those of [9].

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