

Sharing of Knowledge and Preferences among Imperfect Bayesian Participants

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Decision Making (DM) with Limited Resources

Closed DM loop

Participant

- limited resources
- domain-specific

Interface

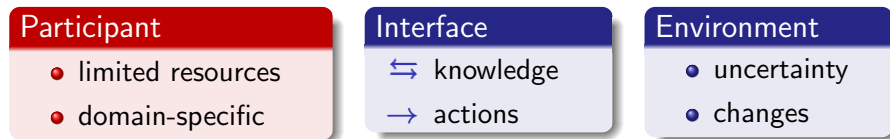
- ↔ knowledge
- actions

Environment

- uncertainty
- changes

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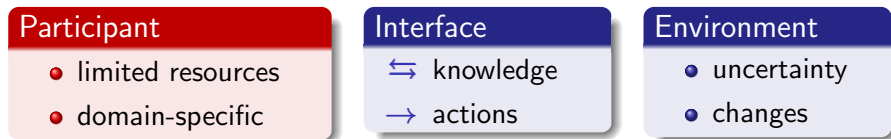


Prescriptive Bayesian decision-making theory treats **DM elements**, i.e., participant's **preferences**, **constraints** and **knowledge** consisting of

- observation of the environment's response on actions
- previously accumulated prior knowledge.

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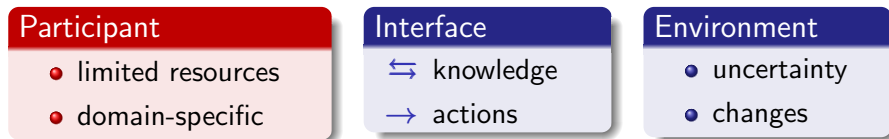
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- map domain-specific DM elements on Bayesian ones?
- share participant's DM elements with others in its environment?

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Limited evaluation resources of the participant have to be respected!

Formalisation of Decision Making

- A participant selects and uses
strategy $s \in \mathbf{s} \equiv \{s : \mathbf{k} \rightarrow a \in \mathbf{a}\}$ to reach a preferred
behaviour $b = [g, a, k] = [\text{ignorance}, \text{action}, \text{knowledge}]$ with
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- **DM under uncertainty** arises unless a participant can uniquely assign b to s . Then **performance index** I_s cannot be optimised and the optimal strategy $!s$ is a minimiser of its expectation $E_s[I_s]$

$$!s \in \text{Arg min}_{s \in \mathbf{s}} E_s[I_s] = \int_{\mathbf{b}} I_s(b) f_s(b) db, \quad (1)$$

where f_s is a Radon-Nikodým derivative (rdn) with respect to a strategy-independent product measure db .

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- f_s describes **closed-loop model** of the participant and its environment, and it holds $f_s(b) = m(b) \times s(b) = \text{environment model} \times \text{strategy}$.

Fully Probabilistic Design (FPD) of Strategy

Optimal strategy l_s implied by the chosen I_s yields **ideal closed-loop model** $l_f(b) = m(b) \times l_s(b) =$ **environment model** \times **optimal strategy**. Thus,

- l_f can be used instead of I_s to describe the preferred behaviour
- absolutely optimal strategy makes closed-loop model f_s equal to l_f
- a strategy providing f_s close to l_f can be taken as the optimal one
- Kullback-Leibler divergence (KLD) measures closeness of f_s to l_f .

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FPD selects the strategy minimising KLD of f_s on l_f

$$o_s \in \text{Arg} \min_{s \in \mathcal{S}} D(f_s || l_f) = \int_{\mathbf{b}} f_s(b) \ln \left(\frac{f_s(b)}{l_f(b)} \right) db \quad (2)$$

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Important features

FPDs densely extend the standard Bayesian designs.

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Important features

FPDs densely extend the standard Bayesian designs.

FPD describes knowledge, constraints & preferences by single language!

FPD: from Math to Real Use

FPD operates on rnds, but participant can provide only domain-specific preferences, constraints and knowledge. How to support their conversion?

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DM elements to be specified

- (1) sets of variables forming ignorance \mathbf{g} , action \mathbf{a} and knowledge \mathbf{k}
- (2) the set of strategies \mathbf{s} among which the optimal one $^O\mathbf{s}(b)$ is searched
- (3) the environment model $m(b)$
- (4) the ideal closed-loop model ${}^l\mathbf{f}(b) = {}^l\mathbf{m}(b) \times {}^l\mathbf{s}(b)$.

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Evaluations soon reach complexity boundaries \Rightarrow **distributed solution**
How to support distributed DM within FPD?

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Construction of (1) and (2) is supported by hypotheses testing.
Construction of (3) and (4) as well as support of distributed DM are addressed by solving appropriate **supporting DM tasks**.

DM elements construction

{domain knowledge, preferences} \rightarrow rnd

- enriching of knowledge on rnd
- approximation of known rnd

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Distributed DM

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Supporting DM Tasks

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Distributed DM

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The supporting DM tasks are formulated and solved via FPD.

Supporting DM Tasks: Notations

- DM elements of supporting DM tasks are denoted by capital letters while that of supported DM task by lower-case letters.

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- A finite cardinality $|\mathbf{b}|$ of the behaviour set $\mathbf{b} = \{b_1, \dots, b_{|\mathbf{b}|}\}$ of the supported DM is assumed. This implies the inspected rnds f be finite-dimensional vectors

$$f \in \mathbf{f} \subset \Delta = \left\{ f(b) : f(b) \geq 0, \int_{b \in \mathbf{b}} f(b) db = 1 \right\} \quad (3)$$

Approximating of a Known Rnd f by \hat{f}

Behaviour $B = [G, A, K] = [\text{ignorance}, \text{action}, \text{knowledge}]$

Part

ignorance b

action \hat{f}

knowledge f

Approximating of a Known Rnd f by \hat{f}

Behaviour $B = [G, A, K] = [\text{ignorance}, \text{action}, \text{knowledge}]$

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Meaning	original behaviour	approximating rnd	approximated rnd

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	Closed-loop model $F(B) = F(b, \hat{f}, f)$		
Factors	$F(b \hat{f}, f)$	$F(\hat{f} f)$	$F(f)$

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Meaning	wish to model b	Left To the Fate	LTF

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$$\hat{f} \in \text{Arg} \min_{\hat{f} \in \hat{F}} D(f||\hat{f}) \dots \textit{information criterion recovered}$$

Enriching of Guess f_0 of Unknown f by Domain Knowledge

Behaviour $B = [G, A, K] = [(b, f), F(f|K), K]$

Part	ignorance $b, f(b)$	action $A = F(f K)$	knowledge $K = \mathbf{f}$
Meaning	behaviour, its rnd	description of f	set of f s

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$f_0 \in \text{Arg min}_{f \in \mathbf{f}} D(f||f_0) \quad \dots \quad \text{minimum KLD principle recovered}$
 $\Leftrightarrow \quad \text{maximum entropy principle for uniform } f_0$

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Choice	$f(b)$	$F_0(f)$	$S(A K)$	$F(K)$
Meaning	model of b	prior guess of A	Left To the Fate	LTF

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Meaning	model of b	action	strategy	K 's rnd

Ideal closed-loop model ${}^I F(B)$

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Meaning	model of b	prior guess of A	Left To the Fate	LTF

- FPD generalises the minimum KLD principle

$${}^O F(f|K) \in \text{Arg} \min_{F \in \mathcal{F}} \int_f F(f) \ln \left(\frac{F(f)}{F_0(f)} \right) df$$

Enriching of Prior Rnd F_0 of an Unknown f by Rnds' Set

Behaviour $B = [G, A, K] = [(b, f), F(f|K), K]$

Part	ignorance $b, f(b)$	action $A = F(f K)$	$K = \mathbf{f}$
Meaning	behaviour, its rnd	description of f	set of f s

Closed-loop model $F(B) = F(b, f, A|K)F(K)$

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DM elements construction

$\{\text{domain knowledge, preferences}\} \rightarrow \text{rnd}$

- enriching of knowledge on rnd
- approximation of known rnd

Distributed DM

$\{\text{rnd}_{\mathcal{K}}\}_{\mathcal{K} \in \mathcal{K}} \rightarrow \text{rnd}$

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$$E_f[\phi_{\kappa}] = \int_{\mathbf{b}} \phi_{\kappa}(b) f(b) db = 0, \quad \kappa \in \boldsymbol{\kappa} = \{1, 2, \dots, |\boldsymbol{\kappa}|\}, \quad |\boldsymbol{\kappa}| < \infty, \quad (\star).$$

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Note: The construction was used for [knowledge elicitation](#). Similarly it can be applied to the ideal rnd construction, i.e., to the [preference elicitation](#).

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Distributed DM: Merging of Probabilistic Knowledge

- Let set \mathbf{f} contain possible compromises between several given rnds $\{f_\kappa(b)\}_{\kappa \in \mathcal{K}} \in \mathbf{\Delta}$ and $F_0(\mathbf{f})$ be the prior rnd on \mathbf{f} .
- The KLD $D(f_\kappa || \mathbf{f})$ specifies how well $f \in \mathbf{f}$ approximates f_κ , i.e., the acceptability of f as the compromise for the κ -th participant. Thus, given thresholds $\{\beta_\kappa\}_{\kappa \in \mathcal{K}}$ specify the meaningful knowledge on f

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$$F_0(\mathbf{f}|K) = \mathcal{D}[\nu_0] \propto \prod_{b \in \mathbf{b}} f(b)^{\nu_0(b)-1} \text{ with } \nu_0(b) > 0, \int_{\mathbf{b}} \nu_0(b) db < \infty$$

gives ${}^{\circ}F(\mathbf{f}) = \mathcal{D}[\nu_0 + \sum_{\kappa \in \mathcal{K}} \lambda_\kappa f_\kappa]$, where $\lambda_\kappa \geq 0$ are Kuhn-Tucker multipliers and $\hat{\mathbf{f}}(b) = E[\mathbf{f}(b)|K]$ is affine combination of $\{f_\kappa\}_{\kappa \in \mathcal{K}}$.

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Merging allows imperfect participants cooperate by sharing “personal” rnds.

Extension of Fragmental Knowledge for Merging

- Imperfect participant provides **fragmental knowledge** or preferences leading to a rnd $f(m_{\kappa}|k_{\kappa})$ derived from $f_{\kappa}(b)$ with

$$\begin{aligned} b &= [u_{\kappa}, m_{\kappa}, k_{\kappa}] = \text{behaviour split to parts} & (5) \\ &= [\text{uninteresting for, modelled by, known to}] \quad \kappa\text{th rnd provider} \end{aligned}$$

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- The extension should be the best approximation of the merger. This together with the merging formula gives

$$\hat{f}(b) = \frac{\nu_0(b) + \sum_{\kappa \in \mathcal{K}} \lambda_\kappa \hat{f}(u_\kappa|m_\kappa, k_\kappa) f_\kappa(m_\kappa|k_\kappa) \hat{f}(k_\kappa)}{\int_{\mathbf{b}} \nu_0(b) db + \sum_{\kappa \in \mathcal{K}} \lambda_\kappa}$$

- The merger is projected back to $(\mathbf{m}_\kappa, \mathbf{k}_\kappa)$ of cooperating imperfect participants. This corrects **DM elements understandable** to them.

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- Can we
 - learn from nature/society something radically different from the approach?
 - use the approach for **modelling** natural/societal systems?