

NON-LOCAL PDES
WITH DISCRETE STATE-DEPENDENT DELAYS:
WELL-POSEDNESS IN A METRIC SPACE

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ABSTRACT. Partial differential equations with discrete (concentrated) state-dependent delays are studied. The existence and uniqueness of solutions with initial data from a wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used to construct a dynamical system. This subset is an analogue of the *solution manifold* proposed for ordinary equations in [H.-O. Walther, The solution manifold and C^1 -smoothness for differential equations with state-dependent delay, J. Differential Equations, 195(1), (2003) 46–65]. The existence of a compact global attractor is proven. As far as applications are concerned, we consider the well known Mackey-Glass-type equations with diffusion, the Lasota-Ważewska-Czyżewska model, and the delayed diffusive Nicholson’s blowflies equation, all with *state-dependent* delays.

1. Introduction. The partial differential equations (PDEs) with delays have attracted a lot of attention during the last decades as many processes of the real world (like an automatically controlled furnace, bi-directional associative memory (BAM) neural networks, reaction-diffusion processes) can be described by such kind of equations. Studying these equations is based on the well-developed approaches to the ordinary differential equations (ODEs) with delays [14, 8, 1] and PDEs without delays [11, 12, 19, 18]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [2, 35, 7] and references therein; see also [36, 5, 6, 3] and to the monograph [43] that are very close to this work.

In many evolution systems arising in applications the presented delays are frequently *state-dependent* (SDDs). The theory of such equations, especially the ODEs, is rapidly developing and many deep results have been obtained up to now (see e.g. [37, 38, 39, 20, 22, 40] and also the survey paper [15] for details and references). The underlying main mathematical difficulty of the theory of PDEs with SDDs lies in the fact that the functions describing state-dependent delays are

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not Lipschitz continuous on the space of continuous functions - the main space, on which the classical theory of equations with delays is developed. This implies that the corresponding *initial value problem* (IVP) is not generally well-posed in the sense of J. Hadamard [11, 12].

The partial differential equations with state-dependent delays were first studied in [26] (the case of distributed delays, weak solutions), [16] (mild solutions, infinite discrete delay), and [27] (weak solutions, finite discrete and distributed delays). An alternative approach to the PDEs with discrete SDDs is proposed in [29].

This paper is a continuation of the work [30] and its goal is to study the approach used for ODEs with SDDs [37, 38, 15] in the case of PDEs. The main idea lies in finding a wider space $Y \supset X$ such that a solution $u : [a, b] \rightarrow Y$ be a Lipschitz function (with respect to a weaker norm of Y), and constructing a dynamical system on a subset of the space $C([a, b]; Y)$. It should be emphasized that the dynamical system is constructed on a metric space that is nonlinear. More precisely, the existence and uniqueness of solutions with initial data from that wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used for constructing the aforementioned dynamical system. This subset is an analogue of *the solution manifold* proposed in [38], see also [15]. We use the same class of non-local in space variables nonlinear PDEs as in [30].

As far as applications and motivations are concerned we consider the well known Mackey-Glass-type equations with diffusion (a model in physiology), the Lasota-Ważewska-Czyżewska model in hematology and the delayed diffusive Nicholson's blowflies equation in population dynamics. The approach proposed here allows to study the models with *state-dependent* delays, which seems to be more realistic comparing with the constant delay. See examples and remark 1 below for more details.

The paper is organized as follows. The section 2 is devoted to the formulation of the model and examples. The proof of the existence and uniqueness of (strong) solutions for initial functions from a Banach space forms a main part of the section 3. In the section 4, an evolution operator S_t is constructed and its asymptotic properties in different functional spaces are investigated. The dissipativeness is obtained in a Banach space, while the existence of a global attractor is proven on a smaller metric space (the solution manifold). The choice of this smaller space is different from that proposed in [30].

2. The model with discrete state-dependent delay and preliminaries.

Consider the following non-local partial differential equation with a discrete state-dependent delay η

$$\frac{\partial}{\partial t} u(t, x) + Au(t, x) + du(t, x) = b([Bu(t - \eta(u_t), \cdot)](x)) \equiv (F_1(u_t))(x), \quad x \in \Omega, \quad (1)$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \rightarrow L^2(\Omega)$ generates an analytic semigroup, $\Omega \subset \mathbb{R}^{n_0}$ is a smooth bounded domain, $B : L^2(\Omega) \rightarrow L^2(\Omega)$ denotes a bounded operator that will be defined later, $b : \mathbb{R} \rightarrow \mathbb{R}$ stands for a Lipschitz map, $d \in \mathbb{R}$, $d \geq 0$, and the function $\eta : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset \mathbb{R}_+$ denotes a *state-dependent discrete delay*. Let $C \equiv C([-r, 0]; L^2(\Omega))$. Norms defined

on $L^2(\Omega)$ and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$, respectively, and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\Omega)$. As usually, $u_t \equiv u_t(\theta) \equiv u(t + \theta)$ for $\theta \in [-r, 0]$.

Examples. As the first application we propose the delayed diffusive equation (see (1)) with Mackey-Glass type nonlinearity given by $b(v) = \beta v(1 + v^m)^{-1}$, $\beta \in \mathbb{R}$, $m \geq 1$. It is the diffusive model of Hematopoiesis (blood cell production) with the state-dependent delay and $-A$ being the Laplace operator with the Neumann boundary conditions. The model of Hematopoiesis when u does not depend on the spatial variable $x \in \Omega$ was first proposed by Mackey and Glass [21]. For the case of constant delay $\eta(u_t) \equiv \tau > 0$ see, e.g. [9, 41] and also [44] for $n_0 = 1$ and further references. One can easily check that the nonlinearity $b(v) = \beta v(1 + v^m)^{-1}$ is bounded and globally Lipschitz provided $m = 2k, k \in \mathbb{N}$.

As another application we can consider the diffusive Nicholson blowflies equation with state-dependent delays [30], i.e. the equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^{n_0}$ is a bounded domain with a smooth boundary, the nonlinear (birth) function b is given by $b(v) = p \cdot v e^{-v}$. For the constant delay see, e.g. [34] and also [44] for $n_0 = 1$.

The third application is the Lasota-Wazewska-Czyzewska model in hematology covering problems involving blood cell pathologies, where the nonlinearity is given by $b(v) = p \cdot v^k e^{-v}$, $p > 0, k \geq 1$. For the case of constant delay ODE see, e.g. [17]. In our case the operator A is as in the Mackey-Glass equation.

Remark 1. The operator B may for example be of the following forms (linear operators)

$$[Bv](x) \equiv \int_{\Omega} v(y) \tilde{f}(x, y) dy, \quad x \in \Omega, \tag{2}$$

or even simpler

$$[Bv](x) \equiv \int_{\Omega} v(y) f(x - y) \ell(y) dy, \quad x \in \Omega, \tag{3}$$

where $f : \Omega \rightarrow \mathbb{R}$ is a smooth function and $\ell \in C_0^\infty(\Omega)$. In the last case the nonlinear term in (1) is of the (nonlocal) form

$$(F_1(u_t))(x) \equiv b \left(\int_{\Omega} u(t - \eta(u_t), y) f(x - y) \ell(y) dy \right), \quad x \in \Omega. \tag{4}$$

□

It is interesting to mention that the presence of the operator B in (1) (in our example it is a nonlocal operator) has not only mathematical but also strong biological motivations as discussed in [4] (see also a survey [10] for relevant references).

Consider the equation (1) with the initial condition

$$u|_{[-r, 0]} = \varphi \tag{5}$$

and let

$$H \equiv \left\{ \varphi \in C([-r, 0]; D(A^{-\frac{1}{2}})) \mid \varphi(0) \in D(A^{\frac{1}{2}}) \right\}. \tag{6}$$

Let further

$$\|\varphi\|_H \equiv \max_{s \in [-r, 0]} \|A^{-\frac{1}{2}} \varphi(s)\| + \|A^{\frac{1}{2}} \varphi(0)\|$$

be a norm defined on the space H and $D(A^\alpha)$ denote the domain of the operator A^α . In the sequel the following assumptions will play an important role.

(H1. η) The discrete delay function $\eta : H \rightarrow [0, r]$ is such that

$$\exists L_\eta > 0, \quad \exists q \geq 0 \text{ such that } \forall \varphi, \psi \in H \Rightarrow$$

$$|\eta(\varphi) - \eta(\psi)| \leq L_\eta \left(q \|A^{\frac{1}{2}}(\varphi(0) - \psi(0))\|^2 + \int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\|^2 d\theta \right)^{\frac{1}{2}} \quad (7)$$

(H.B) The following Lipschitz property of the operator B holds.

$$\exists L_B > 0 \text{ such that } \forall u, v \in D(A^{-\frac{1}{2}}) \Rightarrow \|Bu - Bv\| \leq L_B \|A^{-\frac{1}{2}}(u - v)\| \quad (8)$$

Remark 2. The assumption (8) concerning the operator B implies that B is compact (the case where B is the identity is excluded). This smoothing effect of the operator B is important in our study of discrete state-dependent delays. As a natural example, under the assumption that for all (almost all) $x \in \Omega \Rightarrow f(\cdot - x)\ell(\cdot) \in D(A^{\frac{1}{2}})$ and $u \in D(A^{-\frac{1}{2}})$, the term of the form (3) (a convolution) implies that

$$|\langle u, f(\cdot - x)\ell(\cdot) \rangle| \leq \|A^{-\frac{1}{2}}u\| \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|,$$

which gives

$$\left(\int_{\Omega} \left| \int_{\Omega} u(y)f(y-x)\ell(y)dy \right|^2 dx \right)^{\frac{1}{2}} \leq \|A^{-\frac{1}{2}}u\| \left(\int_{\Omega} \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|^2 dx \right)^{\frac{1}{2}}.$$

Hence, the property (H.B) (see (8)) holds with $L_B \equiv \left(\int_{\Omega} \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|^2 dx \right)^{\frac{1}{2}}$.

The same arguments hold (with $L_B \equiv \left(\int_{\Omega} \|A^{\frac{1}{2}}\tilde{f}(x, \cdot)\|^2 dx \right)^{\frac{1}{2}}$) for a more general term of the form (2).

The same motivation (a smoothing effect due to a nonlocal term) is used in the integral term in (7). The same convolution type example, as just discussed, can serve as a natural example of this effect, i.e. for any Lipschitz function $g : R \rightarrow [0, 1]$ consider

$$\eta(\varphi) = \int_{-r}^0 g \left(\int_{\Omega} \varphi(\theta, y)\ell(y) dy \right) d\theta.$$

One can easily check that

$$\begin{aligned} |\eta(\varphi) - \eta(\psi)| &\leq L_g \int_{-r}^0 |\langle \varphi(\theta) - \psi(\theta), \ell \rangle| d\theta \leq L_g \int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\| \cdot \|A^{\frac{1}{2}}\ell\| d\theta \\ &\leq L_\eta \left(\int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\|^2 d\theta \right)^{\frac{1}{2}}, \quad \text{with } L_\eta \equiv L_g \|A^{\frac{1}{2}}\ell\| r^{\frac{1}{2}}. \end{aligned}$$

From the point of view of applications (the biological ones mentioned in the introduction) it is also natural to have the nonlinearity and delay function of the nonlocal type (for a detailed discussion of the nonlocality of the delay terms see, e.g. [4] and a survey [10] and references therein). \square

Let now the following space

$$\mathcal{L} \equiv \left\{ \varphi \in C([-r, 0]; D(A^{-\frac{1}{2}})) \mid \sup_{s \neq t} \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} \right\} < +\infty; \varphi(0) \in D(A^{\frac{1}{2}}) \right\}, \quad (9)$$

with the natural norm

$$\|\varphi\|_{\mathcal{L}} \equiv \max_{s \in [-r, 0]} \|A^{-\frac{1}{2}}\varphi(s)\| + \sup_{s \neq t} \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} \right\} + \|A^{\frac{1}{2}}\varphi(0)\| \quad (10)$$

be defined. For any segment $[a, b] \subset \mathbb{R}$ (c.f. (9)) and any Lipschitz-on- $[a, b]$ function φ , let

$$\|\varphi\|_{[a, b]} \equiv \sup \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} : s \neq t; s, t \in [a, b] \right\} \quad (11)$$

denote its Lipschitz constant and let $\|\varphi\| \equiv \|\varphi\|_{[-r, 0]}$. Then the following lemma holds.

Lemma 2.1. *Let the assumptions (H1.η) and (H.B) hold (see (7), (8)) and let the function $b : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz. Then any two functions $\varphi \in \mathcal{L}, \psi \in H$ (with H and \mathcal{L} defined in (6) and (9)) the nonlinearity F satisfies*

$$\|F_1(\varphi) - F_1(\psi)\| \leq L_{F_1} \left[\|\varphi\| \right] \left(q \|A^{1/2}(\varphi(0) - \psi(0))\| + \|A^{-1/2}(\varphi - \psi)\|_C \right), \quad (12)$$

where

$$L_{F_1}[\ell] \equiv L_b L_B \sqrt{2} \max \{1; \ell L_\eta \max\{1; \sqrt{r}\}\} \quad (13)$$

and $L_{F_1}[\ell]$ is used in (12) with

$$\ell = L_\varphi \equiv \|\varphi\| \equiv \sup \left\{ \frac{\|A^{-1/2}(\varphi(s) - \varphi(t))\|}{|s - t|} : s \neq t; s, t \in [-r, 0] \right\}.$$

Proof of Lemma 2.1. Using the Lipschitz property of b and B (see (H.B)), it follows that

$$\begin{aligned} \|F_1(\varphi) - F_1(\psi)\|^2 &= \int_{\Omega} |b([B\varphi](-\eta(\varphi), x)) - b([B\psi](-\eta(\psi), x))|^2 dx \leq \\ &\leq L_b^2 \int_{\Omega} |[B\varphi](-\eta(\varphi), x) - [B\psi](-\eta(\psi), x)|^2 dx \\ &= L_b^2 \|[B\varphi](-\eta(\varphi), \cdot) - [B\psi](-\eta(\psi), \cdot)\|^2 \leq \\ &\leq L_b^2 L_B^2 \|A^{-1/2} \{\varphi(-\eta(\varphi)) - \psi(-\eta(\psi)) \pm \varphi(-\eta(\psi))\}\|^2 \leq \\ &\leq 2L_b^2 L_B^2 \left(\|A^{-1/2} \{\varphi(-\eta(\varphi)) - \varphi(-\eta(\psi))\}\|^2 + \|A^{-1/2}(\varphi - \psi)\|_C^2 \right). \end{aligned}$$

Next, $\varphi \in \mathcal{L}$ implies that there exists $L_\varphi \equiv \|\varphi\| > 0$, (see (10),(11)) such that

$$\|A^{-1/2}(\varphi(s^1) - \varphi(s^2))\| \leq L_\varphi |s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, 0]. \quad (14)$$

Hence, (14) and (H1.η) give

$$\|F_1(\varphi) - F_1(\psi)\|^2 \leq$$

$$\begin{aligned}
&\leq 2L_b^2 L_B^2 \left[L_\varphi^2 L_\eta^2 \left(q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + \int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\|^2 d\theta \right) + \right. \\
&\quad \left. + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right] \leq \\
&\leq 2L_b^2 L_B^2 \left[L_\varphi^2 L_\eta^2 \left(q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + r \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right) + \right. \\
&\quad \left. + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right] \leq \\
&\leq 2L_b^2 L_B^2 \max\{1; L_\varphi^2 L_\eta^2 \max\{1; r\}\} \left[q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right].
\end{aligned}$$

The last estimate and using the formulas $\sqrt{\max\{|a|; |b|\}} = \max\{\sqrt{|a|}; \sqrt{|b|\}$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$ give (12), (13), which completes the proof. \square

3. The existence and uniqueness of solutions. As in [30] we need the following

Definition 3.1. A vector-function $u(t) \in C([-r, T]; D(A^{-1/2})) \cap C([0, T]; D(A^{1/2})) \cap L^2(0, T; D(A))$ with derivative $\dot{u}(t) \in L^\infty(0, T; D(A^{-1/2}))$ is a solution to the problem defined by (1) and (5) on $[0, T]$ if

- (a) $u(\theta) = \varphi(\theta)$ for $\theta \in [-r, 0]$;
- (b) $\forall v \in L^2(0, T; L^2(\Omega))$ such that $\dot{v} \in L^2(0, T; D(A^{-1}))$ and $v(T) = 0 \Rightarrow$

$$\begin{aligned}
& - \int_0^T \langle u(t), \dot{v}(t) \rangle dt + \int_0^T \langle A^{1/2}u(t), A^{1/2}v(t) \rangle dt = \\
& = \langle \varphi(0), v(0) \rangle + \int_0^T \langle F_1(u_t) - d \cdot u(t), v(t) \rangle dt. \quad (15)
\end{aligned}$$

Now we prove the following theorem on the existence and uniqueness of solutions.

Theorem 3.2. Let the assumptions (H1. η) and (H.B) hold and let the function $b : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz and bounded, i.e. $|b(s)| \leq M_b$ for all $s \in \mathbb{R}$. Let further $\varphi \in \mathcal{L}$ be a given initial condition. Then the problem defined by (1) and (5) has a unique solution on any time interval $[0, T]$ such that $\dot{u} \in L^2(0, T; L^2(\Omega))$.

Remark 3. Notice that φ does not assume $\varphi \in L^2([-r, 0]; D(A))$. However, the definition of a solution implies that

$$u_t \in L^2([-r, 0]; D(A)), \quad \forall t \geq r. \quad (16)$$

\square

Proof. We follow the proof of Theorem 1 in [30]. Notice that the assumption (H1. η) is slightly more general than the assumption (H. η) in [30]. This implies some changes in the proof of the uniqueness of solutions.

Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis of $L^2(\Omega)$ such that $Ae_k = \lambda_k e_k$, $0 < \lambda_1 < \dots < \lambda_k \rightarrow +\infty$ and consider the Galerkin approximate solution $u^m = u^m(t, x) = \sum_{k=1}^m g_{k,m}(t)e_k$ of order m such that

$$\begin{cases} \langle \dot{u}^m + Au^m + du^m - F_1(u_t^m), e_k \rangle = 0, \\ \langle u^m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle, \quad \forall \theta \in [-r, 0] \end{cases} \quad (17)$$

$\forall k = 1, \dots, m, g_{k,m} \in C^1(0, T; \mathbb{R}) \cap L^2(-r, T; \mathbb{R})$ with $g_{k,m}(t)$ absolutely continuous.

The system (17) is a system of (ordinary) differential equations in \mathbb{R}^m with a concentrated (discrete) state-dependent delay for the unknown vector function $U(t) \equiv (g_{1,m}(t), \dots, g_{m,m}(t))$ (for the corresponding theory see [38, 39] and also a recent review [15]).

The key difference between equations with state-dependent and state-independent (concentrated) delays is that the first type of equations is not well-posed in the space of continuous (initial) functions. To get a well-posed initial value problem, it is better [38, 39, 15] to use a smaller space of Lipschitz continuous functions or even a smaller subspace of $C^1([-r, 0]; \mathbb{R}^m)$.

The condition $\varphi \in \mathcal{L}$ implies that the function $U(\cdot)|_{[-r,0]} \equiv P_m \varphi(\cdot)$, which defines initial data, is Lipschitz continuous as a function from $[-r, 0]$ to \mathbb{R}^m . Here P_m is the orthogonal projection onto the subspace $span \{e_1, \dots, e_m\} \subset L^2(\Omega)$. Hence, we can apply the theory of ODEs with discrete state-dependent delay (see e.g. [15]) to get the local existence and uniqueness of solutions to (17).

Next, we will get an a priori estimate to prove the continuation of solutions u^m to (17) on any time interval $[0, T]$ and then use it for the proof (by the method of compactness, see [19]) of the existence of strong solutions to (1) and (5). To that end, multiply the first equation in (17) by $\lambda_k g_{k,m}$ and sum for $k = 1, \dots, m$ to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{1/2} u^m(t)\|^2 + \|Au^m(t)\|^2 + d \cdot \|A^{1/2} u^m(t)\|^2 &= \langle P_m F(u_t^m), Au^m(t) \rangle \leq \\ &\leq \frac{1}{2} \|P_m F(u_t^m)\|^2 + \frac{1}{2} \|Au^m(t)\|^2. \end{aligned}$$

As the function b is bounded, $\|F(u_t^m)\|^2 \leq M_b^2 |\Omega|$ (here $|\Omega| \equiv \int_{\Omega} 1 \, dx$), which gives

$$\frac{d}{dt} \|A^{1/2} u^m(t)\|^2 + \|Au^m(t)\|^2 \leq M_b^2 |\Omega|. \tag{18}$$

Integrating (18) with respect to t and using the relationships $\varphi(0) \in D(A^{1/2})$, $u^m(0) = P_m \varphi(0) \in D(A^{1/2})$, $\|A^{1/2} u^m(0)\| = \|A^{1/2} P_m \varphi(0)\| \leq \|A^{1/2} \varphi(0)\|$, we get an a priori estimate

$$\|A^{1/2} u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2 \, d\tau \leq \|A^{1/2} \varphi(0)\|^2 + M_b^2 |\Omega| T, \quad \forall m, \forall t \in [0, T]. \tag{19}$$

The above relationship (19) means that

$$\{u^m\}_{m=1}^{\infty} \text{ is a bounded set in } L^{\infty}(0, T; D(A^{1/2})) \cap L^2(0, T; D(A)).$$

Using this fact and (17), it follows that

$$\{\dot{u}^m\}_{m=1}^{\infty} \text{ is a bounded set in } L^{\infty}(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega)).$$

Hence, the family $\{(u^m; \dot{u}^m)\}_{m=1}^{\infty}$ is a bounded set in

$$\begin{aligned} Z_1 \equiv &\left(L^{\infty}(0, T; D(A^{1/2})) \cap L^2(0, T; D(A)) \right) \times \\ &\times \left(L^{\infty}(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega)) \right). \end{aligned} \tag{20}$$

Therefore, there exist a subsequence $\{(u^k; \dot{u}^k)\}$ and an element $(u; \dot{u}) \in Z_1$ such that

$$\{(u^k; \dot{u}^k)\} \text{ *-weak converges to } (u; \dot{u}) \text{ in } Z_1. \tag{21}$$

The proof that any $*$ -weak limit is a strong solution is standard. To prove the property $u(t) \in C([0, T]; D(A^{1/2}))$, we use the well-known (see also [18, thm. 1.3.1])

Theorem 3.3. (Proposition 1.2 in [31]). *Let V denote a dense Banach space that is continuously embedded in a Hilbert space X and let $X = X^*$ so that $V \hookrightarrow X \hookrightarrow V^*$. Then the Banach space $W_p(0, T) \equiv \{u \in L^p(0, T; V) : \dot{u} \in L^q(0, T; V^*)\}$ (here $p^{-1} + q^{-1} = 1$) is contained in $C([0, T]; X)$.*

In our case $X = D(A^{1/2})$, $V = D(A)$, $V^* = L^2(\Omega)$, $p = q = 1/2$.

Now we prove the uniqueness of solutions. Using the fact that $\varphi \in \mathcal{L}$, the definition 3.1 of a solution v , and $\dot{v}(t) \in L^\infty(0, T; D(A^{-1/2}))$ (see (21)), it follows that for any such a solution v and any $T > 0$, there exists $L_{v,T} > 0$, such that

$$\|A^{-1/2}(v(s^1) - v(s^2))\| \leq L_{v,T}|s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, T]. \tag{22}$$

In the light of (11), let $L_{v,T} \equiv \|v\|_{[-r, T]}$.

Consider any two solutions u and v of (1), (5) (not necessarily with the same initial function). The standard variation-of-constants formula $u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-\tau)}F(u_\tau) d\tau$ and the estimate $\|A^\alpha e^{-tA}\| \leq (\frac{\alpha}{t})^\alpha e^{-\alpha}$ (see e.g. [7, (1.17), p.84]) give

$$\begin{aligned} & \|A^{1/2}(u(t) - v(t))\| \\ & \leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \int_0^t \|A^{1/2}e^{-A(t-\tau)}\| \|F(u_\tau) - F(v_\tau)\| d\tau \\ & \leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \int_0^t \left(\frac{1/2}{t-\tau}\right)^{1/2} e^{-1/2} \|F(u_\tau) - F(v_\tau)\| d\tau, \end{aligned} \tag{23}$$

as $\|A^{1/2}e^{-A(t-\tau)}\| \leq (\frac{1/2}{t-\tau})^{1/2} e^{-1/2}$ and similarly,

$$\|A^{-1/2}(u_t - v_t)\|_C \leq \|A^{-1/2}(u_0 - v_0)\|_C + \int_0^t \|F(u_\tau) - F(v_\tau)\| d\tau.$$

The last estimate and (23) give (just the case when $q = 1$ is shown for the purpose of clarity)

$$\begin{aligned} & \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C \leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \\ & + \|A^{-1/2}(u_0 - v_0)\|_C + \int_0^t \left\{1 + (2e(t-\tau))^{-1/2}\right\} \|F(u_\tau) - F(v_\tau)\| d\tau. \end{aligned} \tag{24}$$

It follows, from Lemma 2.1, that

$$\|F(u_t) - F(v_t)\| \leq L_{F_1, v, T} \left(q \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C \right), \tag{25}$$

where $L_{F_1, v, T}$ is defined in the same way as L_{F_1} in (13), just with $\ell = L_{v, T}$ instead of L_φ - see (13) and (22).

$$L_{F_1, v, T} \equiv L_{F_1} \left[L_{v, T} \right] \equiv L_b L_B \sqrt{2} \max \{1; L_{v, T} L_\eta \max\{1; \sqrt{r}\}\}. \tag{26}$$

It should be emphasized how the Lipschitz constant $L_{v, T} \equiv \|v\|_{[-r, T]}$ of a strong solution v is taken into account in (26) (see (22) and (11)).

Let

$$g(t) \equiv \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C.$$

Then the relationships (24) and (25) lead to the following estimate

$$g(t) \leq g(0) + \int_0^t \left\{ 1 + (2e(t - \tau))^{-1/2} \right\} L_{F_1, v, T} \cdot g(\tau) d\tau$$

Lemma 3.4 (Gronwall). *Let $u, \alpha \in C[a, b], \beta(t) \geq 0, \beta$ is integrable on $[a, b]$ and*

$$u(t) \leq \alpha(t) + \int_a^t \beta(\tau)u(\tau) d\tau, \quad a \leq t \leq b$$

Then

$$u(t) \leq \alpha(t) + \int_a^t \beta(\tau)\alpha(\tau) \exp \left\{ \int_\tau^t \beta(s) ds \right\} d\tau, \quad a \leq t \leq b$$

Moreover, if α is non-decreasing, then

$$u(t) \leq \alpha(t) \exp \left\{ \int_a^t \beta(s) ds \right\}, \quad a \leq t \leq b.$$

It follows, from the above lemma and equality $\int_0^t (t - \tau)^{-1/2} d\tau = 2t^{1/2}$, that

$$\begin{aligned} g(t) &\leq g(0) \exp \left\{ L_{F_1, v, T} \int_0^t \left\{ 1 + (2e(t - s))^{-1/2} \right\} ds \right\} \leq \\ &\leq g(0) \exp \left\{ L_{F_1, v, T} \left(t + \sqrt{\frac{2t}{e}} \right) \right\}, \end{aligned}$$

which implies, $\forall t \in [0, T]$, that

$$\begin{aligned} \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C &\leq \\ &\leq E_{F_1, v, T} \left(\|A^{1/2}(u(0) - v(0))\| + \|A^{-1/2}(u_0 - v_0)\|_C \right), \end{aligned} \quad (27)$$

where

$$E_{F_1, v, T} \equiv \exp \left\{ L_{F_1, v, T} \cdot \left(T + \sqrt{\frac{2T}{e}} \right) \right\}, \quad (28)$$

see (26) for the definition of $L_{F_1, v, T} \equiv L_{F_1} [L_{v, T}]$. This proves the uniqueness of the solution to (1) and (5), and completes the proof of the theorem. \square

4. Asymptotic properties of solutions. This section is devoted to studies of the asymptotic behavior of solutions in different functional spaces. We define first (in a standard way) the evolution semigroup $S_t : \mathcal{L} \rightarrow \mathcal{L}$ (the space \mathcal{L} is defined in (9)) by the formula

$$S_t \varphi \equiv u_t, \quad t \geq 0, \quad (29)$$

where $u(t)$ is a unique solution to the problem (1) and (5) (see definition 3.1).

The estimate (27) means the continuity of the evolution operator S_t in the norm of the space H (see (6)), i.e.

$$\|S_t \varphi - S_t \psi\|_H \leq E_{F_1, v, T} \|\varphi - \psi\|_H \text{ for all } t \in [0, T]. \quad (30)$$

The aim now is to get a more precise estimate, e.g. the continuity of S_t in the norm of the space \mathcal{L} (see (9), (10)). Consider the definition of the Galerkin approximate solution (see (17)). It gives

$$\|A^{-1/2}(\dot{u}^m(t) - \dot{v}^m(t))\| \leq \|A^{1/2}(u^m(t) - v^m(t))\| + d\|A^{-1/2}(u^m(t) - v^m(t))\| + \|F_1(u_t^m) - F_1(v_t^m)\|$$

and Lemma 2.1 implies

$$\|A^{-1/2}(\dot{u}^m(t) - \dot{v}^m(t))\| \leq (1 + d + L_{F_1})\{\|A^{1/2}(u^m(t) - v^m(t))\| + \|A^{-1/2}(u_t^m - v_t^m)\|_C\}.$$

An analogous estimate for a solution to the problem (1) and (5), can be obtained from (21) and the following

Theorem 4.1. [45, Chapter V, Theorem 9, p.125] *Let X be a Banach space. Then any $*$ -weak convergent sequence $\{w_k\}_{k=1}^\infty \in X^*$ $*$ -weak converges to an element $w_\infty \in X^*$ and $\|w_\infty\|_{X^*} \leq \liminf_{n \rightarrow \infty} \|w_n\|_{X^*}$.*

More precisely,

$$\operatorname{ess\,sup}_{t \in [0, T]} \|A^{-1/2}(\dot{u}(t) - \dot{v}(t))\| \leq (1 + d + L_{F_1}) \sup_{t \in [0, T]} \{\|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C\}$$

The last estimate and relationship (27) imply

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} \|A^{-1/2}(\dot{u}(t) - \dot{v}(t))\| &\leq \\ &\leq (1 + d + L_{F_1})E_{F_1, v, T} \left(\|A^{1/2}(u(0) - v(0))\| + \|A^{-1/2}(u_0 - v_0)\|_C \right) \end{aligned} \quad (31)$$

Hence, see (11),

$$\|u - v\|_{[0, T]} \leq (1 + d + L_{F_1})E_{F_1, v, T} \left(\|A^{1/2}(u(0) - v(0))\| + \|A^{-1/2}(u_0 - v_0)\|_C \right)$$

From that and (27), it follows that

$$\|u_t - v_t\|_{\mathcal{L}} \leq (2 + d + L_{F_1})E_{F_1, v, T} \|u_0 - v_0\|_{\mathcal{L}}, \quad \forall t \in [0, T], \quad (32)$$

which finally means that for any $T \geq 0$ there exists a constant $C_T > 0$ such that $\forall t \in [0, T]$ it gives

$$\|u_t - v_t\|_{\mathcal{L}} = \|S_t \varphi - S_t \psi\|_{\mathcal{L}} \leq C_T \|\varphi - \psi\|_{\mathcal{L}}, \quad \forall \varphi, \psi \in \mathcal{L} \quad (33)$$

The last inequality means the continuity of the evolution operator S_t in the norm of the space \mathcal{L} (see (29) and compare with (30)).

Remark 4. It should be noted that the evolution operator and, more generally, the time-shift is not a (strongly) continuous mapping in the norm of the space \mathcal{L} (see (9)). This can be illustrated by the following simple (scalar) example.

Consider the space

$$\mathcal{L}ip([-r, T]; \mathbb{R}) \equiv \left\{ v : [-r, T] \rightarrow \mathbb{R} : \sup \left\{ \frac{|v(s) - v(t)|}{|s - t|}, s \neq t; s, t \in [-r, T] \right\} < \infty \right\}$$

and analogously define the space $\mathcal{L}ip([-r, 0]; \mathbb{R})$ with the natural norm

$$\|v\|_{\mathcal{L}ip} \equiv \max_{\theta \in [-r, 0]} |v(\theta)| + \sup \left\{ \frac{|v(s) - v(t)|}{|s - t|}, s \neq t; s, t \in [-r, 0] \right\}.$$

The (strong) continuity of the time-shift means that

$$\forall v \in \mathcal{L}ip([-r, T]; \mathbb{R}) \text{ and } \forall t \in [0, T] \implies \lim_{h \rightarrow 0} \|v_{t+h} - v_t\|_{\mathcal{L}ip} = 0. \tag{34}$$

Obviously, when $t = 0$ one considers $h \rightarrow 0^+$, while for $t = T$, the case $h \rightarrow 0^-$ should be investigated.

To prove the claim, we must show that (34) does not hold, i.e.

$$\exists v \in \mathcal{L}ip([-r, T]; \mathbb{R}) \text{ and } \exists t_0 \in [0, T] \text{ for which } \lim_{h \rightarrow 0} \|v_{t_0+h} - v_{t_0}\|_{\mathcal{L}ip} \neq 0. \tag{35}$$

Thus, consider the case $t_0 = 0, h \rightarrow 0^+$ and the function

$$v(t) \equiv \begin{cases} 0, & t \in [-r, 0] \\ t, & t \in (0, T] \end{cases}.$$

It can be seen that $v_{t_0} = v_0 \equiv 0$ and

$$v_{t_0+h} = v_{t_0+h}(\theta) = \begin{cases} 0, & \theta \in [-r, -h] \\ h + \theta, & \theta \in (-h, 0] \end{cases}.$$

Hence, $\|v_{t_0+h} - v_{t_0}\|_{\mathcal{L}ip} = \|v_{t_0+h}\|_{\mathcal{L}ip} = h + 1$ and finally $\lim_{h \rightarrow 0^+} \|v_{t_0+h} - v_{t_0}\|_{\mathcal{L}ip} = \lim_{h \rightarrow 0^+} (h + 1) = 1 \neq 0$, which means that (34) does not hold. In the space \mathcal{L} , we would proceed analogously. \square

Remark 5. In the same way as in the previous remark one can show that the time-shift is **not** a (strongly) continuous mapping in the topology of $L^\infty(-r, 0)$.

One could consider the function $\tilde{v}(t) \equiv \begin{cases} 0, & t \in [-r, 0] \\ 1, & t \in (0, T] \end{cases}$ and $t_0 = 0$ to show that $\lim_{h \rightarrow 0^+} \|\tilde{v}_h - \tilde{v}_0\|_{L^\infty(-r, 0)} = 1 \neq 0$. By the way, $\tilde{v} = \frac{d}{dt}v$, where, as usually, the time-derivative is understood in the sense of distributions. \square

The above remarks show that despite of the existence and uniqueness of solutions in the space \mathcal{L} and even strong continuity of the evolution operator S_t in the norm of \mathcal{L} (see (33)), the pair $(S_t; \mathcal{L})$ does *not* form a *dynamical system* since S_t is not strongly continuous as a mapping of time variable.

The methods, developed for ordinary delay equations in [38] suggest to restrict our considerations to a smaller subset of the space of Lipschitz functions. In this paper we follow this suggestion and consider the evolution operator S_t on the following subset of \mathcal{L}

$$X \equiv \left\{ \varphi \in C^1([-r, 0]; D(A^{-1/2})) \text{ such that } \begin{aligned} & \varphi(0) \in D(A^{1/2}) \text{ and } \dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi) \end{aligned} \right\} \subset \mathcal{L}. \tag{36}$$

Here the equality $\dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi)$ is understood as an equality in $D(A^{-1/2})$.

Remark 6. The set X is an analogue of the *solution manifold* introduced in [38] for the case of ODEs with state-dependent delays. \square

To show that the set X is invariant under the evolution operator S_t , we first have to establish an additional smoothness property of the solutions of problem (1), (5).

Lemma 4.2. For any $\varphi \in C^1([-r, 0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$, the solution to (1), (5) (which is given by Theorem 3.2) has the property (c.f. Theorem 3.3 and Theorem 3.2)

$$\dot{u} \in C([0, T]; D(A^{-1/2})), \quad \forall T > 0. \quad (37)$$

Remark 7. We do not assume $\varphi(0) \in D(A)$, just $\varphi(0) \in D(A^{1/2})$, so we cannot directly use [24, Theorem 3.5, p.114].

Proof of Lemma 4.2. By Theorems 3.3, 3.2, for any $\varphi \in C^1([-r, 0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$, there exists a unique solution $u(t) \in C([-r, T]; D(A^{-1/2})) \cap C([0, T]; D(A^{1/2}))$. This property and Lemma 2.1 then imply the continuity of the function

$$p(t) \equiv F_1(u_t) \in C([0, T]; L^2(\Omega)). \quad (38)$$

Consider the following auxiliary linear system without delay

$$\begin{cases} \dot{v}(t) + Av(t) + dv(t) = p(t), & t \geq 0, \\ v(0) = \varphi(0) \in D(A^{1/2}) \end{cases} \quad (39)$$

In the same way as in (17), the Galerkin approximate solution $v^m = v^m(t, x) = \sum_{k=1}^m g_{k,m}(t)e_k$ of order m to (39) can be defined such that

$$\begin{cases} \langle \dot{v}^m + Av^m + dv^m - p(t), e_k \rangle = 0, & t \geq 0, \\ \langle v^m(0), e_k \rangle = \langle \varphi(0), e_k \rangle, & \forall k = 1, \dots, m. \end{cases} \quad (40)$$

where $g_{k,m} \in C^1(0, T; \mathbb{R}) \cap L^2(-r, T; \mathbb{R})$ and $\dot{g}_{k,m}(t)$ is absolutely continuous.

The difference between approximate solutions u^m and v^m lies in that v^m are solutions just to linear system (40). So, for any two approximate solutions v^n and v^m (solutions to (40) of different orders n and m), one has $g_{k,n}(t) \equiv g_{k,m}(t)$, which is denoted by $g_k(t)$.

Multiply (40) by $\lambda_k g_k$ and sum for $k = n+1, \dots, n+p$ (p is any positive integer) to get

$$\begin{aligned} & \langle \dot{v}^{n+p}(t) - \dot{v}^n(t), A(v^{n+p}(t) - v^n(t)) \rangle + \|A(v^{n+p}(t) - v^n(t))\|^2 + \\ & + d \langle v^{n+p}(t) - v^n(t), A(v^{n+p}(t) - v^n(t)) \rangle = \langle (P_{n+p} - P_n)p(t), A(v^{n+p}(t) - v^n(t)) \rangle \end{aligned}$$

It should be recalled that, see the proof of Theorem 3.2, P_m is the orthogonal projection onto the subspace $\text{span}\{e_1, \dots, e_m\} \subset L^2(\Omega)$. Hence,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \|A(v^{n+p}(t) - v^n(t))\|^2 + d \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 \leq \\ & \leq \|(P_{n+p} - P_n)p(t)\| \cdot \|A(v^{n+p}(t) - v^n(t))\| \leq \frac{1}{2} \|(P_{n+p} - P_n)p(t)\|^2 + \\ & \quad + \frac{1}{2} \|A(v^{n+p}(t) - v^n(t))\|^2 \end{aligned}$$

which gives

$$\frac{d}{dt} \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \|A(v^{n+p}(t) - v^n(t))\|^2 \leq \|(P_{n+p} - P_n)p(t)\|^2.$$

Integrating the last estimate results ($\forall t \in [0, T]$) in

$$\begin{aligned} & \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \int_0^t \|A(v^{n+p}(\tau) - v^n(\tau))\|^2 d\tau \leq \\ & \leq \|A^{\frac{1}{2}}(v^{n+p}(0) - v^n(0))\|^2 + \int_0^t \|(P_{n+p} - P_n)p(\tau)\|^2 d\tau \leq \\ & \leq \|(P_{n+p} - P_n)A^{\frac{1}{2}}\varphi(0)\|^2 + \int_0^T \|(P_{n+p} - P_n)p(\tau)\|^2 d\tau. \end{aligned}$$

Summing up, the above estimate, the fact that $\varphi(0) \in D(A^{1/2})$, the strong convergence $\|I - P_n\| \rightarrow 0$ for $n \rightarrow \infty$, and (38) imply that

$$\text{the sequence } \{v^n\}_{n=1}^\infty \text{ is a Cauchy sequence in } C([0, T]; D(A^{\frac{1}{2}})). \tag{41}$$

Now our goal is to show that the sequence $\{\dot{v}^n\}_{n=1}^\infty$ is a Cauchy sequence in $C([0, T]; D(A^{-1/2}))$. So, multiply first (40) by $\lambda_k^{-\frac{1}{2}}$ to get $\lambda_k^{-\frac{1}{2}}\dot{g}_k(t) = -\lambda_k^{\frac{1}{2}}g_k(t) - d\lambda_k^{-\frac{1}{2}}g_k(t) + \langle \lambda_k^{-\frac{1}{2}}p(t), e_k \rangle$. This gives $\lambda_k^{-1}(\dot{g}_k(t))^2 \leq 3\lambda_k(g_k(t))^2 + 3d^2\lambda_k^{-1}(g_k(t))^2 + 3|\langle \lambda_k^{-\frac{1}{2}}p(t), e_k \rangle|^2$. The sum for $k = n + 1, \dots, n + p$ reads

$$\begin{aligned} & \|A^{-\frac{1}{2}}(\dot{v}^{n+p}(t) - \dot{v}^n(t))\|^2 \leq 3\|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \\ & + 3d^2\|A^{-\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \frac{3}{\lambda_{n+1}}\|(P_{n+p} - P_n)p(t)\|^2 \leq \\ & \leq 3\left(1 + \frac{d^2}{\lambda_{n+1}^2}\right)\|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \frac{3}{\lambda_{n+1}}\|I - P_n\|^2\|p(t)\|^2. \end{aligned}$$

The last estimation together with (41) give that

$$\text{the sequence } \{\dot{v}^n\}_{n=1}^\infty \text{ is a Cauchy sequence in } C([0, T]; D(A^{-\frac{1}{2}})). \tag{42}$$

Thus, there exists a unique solution $v(t)$ ($v \equiv \lim_{n \rightarrow \infty} v^n$) to the linear system (39), which satisfies $v \in C([0, T]; D(A^{\frac{1}{2}}))$ and $\dot{v} \in C([0, T]; D(A^{-\frac{1}{2}}))$.

On the other hand, the nonlinear delay system (1), (5) with the initial function φ has also a unique solution. From the construction of $p(t)$ (see (38)), it follows that $u(t) \equiv v(t)$ for all $t \in [0, T]$, which gives (37) and completes the proof of Lemma 4.2. \square

Lemma 4.2 particularly shows that the set X , defined by (36), is invariant under the evolution operator S_t (see (29)). This fact allows to define an evolution operator (denoted again by S_t) $S_t : X \rightarrow X$ in the same way as in (29). Now, if the natural norm

$$\|\varphi\|_X \equiv \max_{s \in [-r, 0]} \|A^{-1/2}\varphi(s)\| + \max_{s \in [-r, 0]} \|A^{-1/2}\dot{\varphi}(s)\| + \|A^{1/2}\varphi(0)\|$$

on X is taken into account, then Theorem 3.2, Lemma 4.2, and Theorem 3.3 give the continuity of S_t with respect to t in the norm of X . Hence, $(S_t; X)$ defines a dynamical system.

Now we will pay attention to the long-time asymptotic behavior of the constructed evolution semigroup $S_t : X \rightarrow X$.

Theorem 4.3. *Using the above notation and under the assumptions of Theorem 3.2, the dynamical system (S_t, X) is dissipative. If, in addition, $q = 0$ in*

(H1.η), then (S_t, X) possesses a compact global attractor \mathcal{A} , which is a bounded set in the space $C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^\alpha))$, $\alpha \in (\frac{1}{2}, 1)$.

Proof of Theorem 4.3. It will be shown first that (S_t, X) is a dissipative dynamical system. To that end, the below proposition is needed.

Lemma 4.4. [30, Lemma 1] *Let all the assumptions of Theorem 3.2 hold and let $\alpha \in (\frac{1}{2}, 1)$. Then there exists a bounded subset $\mathcal{B}V_\alpha$ of the space $C^1([-r, 0]; D(A^{-\frac{1}{2}})) \cap C([-r, 0]; D(A^\alpha))$, which absorbs any strong solution to the problem (1) and (5) for any initial function $\varphi \in \mathcal{L}$.*

Second, to apply the classical theorem on the existence of a global attractor (see, for example [2, 35, 7]), we show that (S_t, X) is asymptotically compact. Consider therefore any solution $u(t)$ to the problem (1) and (5) with $\varphi \in \mathcal{B}V_\alpha$ as an initial function. We will show that for any $\delta > r > 0$ and any $T > \delta$ the set $\mathcal{U} \equiv \{u_t = S_t\varphi \mid \varphi \in \mathcal{B}V_\alpha, t \in [\delta, T]\}$ is relatively compact in X .

Recall that the set $\mathcal{B}V_\alpha$ is a ball in $C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^\alpha))$ (for more details see [30]) and notice that, by Corollary 4 from [32], the set $\mathcal{B}V_\alpha$ is relatively compact in $C([-r, 0]; D(A^{-1/2}))$ (see also [32, lemma 1]). It remains to show that $\{\dot{u}(t) \mid \varphi \in \mathcal{B}V_\alpha, t \in [\delta - r, T]\}$ is equi-continuous in $C([\delta - r, T]; D(A^{-1/2}))$.

Theorem 4.5. [24, Corollary 4.3.3 and Theorem 4.3.5]. *Let A be an infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$. If $f \in L^1((0, T); Y)$ is locally Hölder continuous on $(0, T]$, then for every $x \in Y$ the initial value problem*

$$\dot{u}(t) = Au(t) + f(t), t > 0; \quad u(0) = x$$

has a unique solution u . If $f \in C^\theta([0, T]; Y)$, then for every $\delta > 0$, $Au \in C^\theta([\delta, T]; Y)$ and $\dot{u} \in C^\theta([\delta, T]; Y)$.

Here $C^\theta([0, T]; Y)$ denotes the family of all Hölder continuous functions on $[0, T]$ with the exponent $\theta \in (0, 1)$. In this case, $Y = L^2(\Omega)$.

In order to apply Theorem 4.5 to our case, we have to show that $p(t) = F_1(u_t) \in C^\theta([\delta - r, T]; L^2(\Omega))$ (c.f. (38), (39)). Therefore, consider $t \in [\delta - r, T]$ and

$$\begin{aligned} \|p(t+h) - p(t)\| &= \|F_1(u_{t+h}) - F_1(u_t)\| \leq \\ &\leq L_{F_1}[\ell_{\mathcal{B}V_\alpha}] \max_{s \in [-r, 0]} \|A^{-1/2}(u(t+h+s) - u(t+s))\| \leq L_{F_1}[\ell_{\mathcal{B}V_\alpha}] \ell_{\mathcal{B}V_\alpha} |h| \end{aligned}$$

where $L_{F_1}[\ell_{\mathcal{B}V_\alpha}]$ is the constant defined in Lemma 2.1 with $\ell_{\mathcal{B}V_\alpha}$ such that $\|\psi\| \leq \ell_{\mathcal{B}V_\alpha} \forall \psi \in \mathcal{B}V_\alpha$ (the existence of such $\ell_{\mathcal{B}V_\alpha}$ follows from Theorem 4.1). Here, $q = 0$ is used.

The last inequality shows that $p : [\delta - r, T] \rightarrow L^2(\Omega)$ is Lipschitz continuous, which is the situation to which Lemma 4.4 can be applied. It should also be noted that the family $\{p(t)\}$, for all initial $\varphi \in \mathcal{B}V_\alpha$, is uniformly Lipschitz, i.e. all the Lipschitz constants are lower or equal to $L \equiv L_{F_1}[\ell_{\mathcal{B}V_\alpha}] \cdot \ell_{\mathcal{B}V_\alpha}$. Then by Theorem 4.1, it is guaranteed (see the proof) that the family $\{\dot{u}(t) \mid \varphi \in \mathcal{B}V_\alpha, t \in [\delta - r, T]\}$ is uniformly Hölder continuous, and thus equi-continuous in $C([\delta - r, T]; L^2(\Omega))$.

Theorem 4.6. [32, lemma 1] *Let B be a Banach space. A set F of $C([0, T]; B)$ is relatively compact if and only if*

$$(i) \quad F(t) \equiv \{f(t) : f \in F\} \text{ is relatively compact in } B, \quad 0 < t < T,$$

- (ii) F is uniformly equicontinuous, i.e. $\forall \varepsilon > 0, \exists \eta$ such that $\|f(t_2) - f(t_1)\|_B \leq \varepsilon, \forall f \in F, \forall 0 \leq t_1 \leq t_2 \leq T$ such that $|t_2 - t_1| \leq \eta$

Applying Theorem 4.6 completes the proof of Theorem 4.3. \square

In all the applications considered above (Mackey-Glass-type equations with diffusion, the Lasota-Ważewska-Czyżewska model in hematology and the delayed diffusive Nicholson's blowflies equation) the functions b are bounded and globally Lipschitz, so for any delay function η satisfying (H1. η) and any (nonlocal) operator B , satisfying (H.B) (as discussed in Remark 2), the conditions of Theorem 3.2 and Theorem 4.3 are satisfied. As a result, we conclude that the initial value problem (1) and (5) is well-posed in X and the dynamical system (S_t, X) has a global attractor (Theorem 4.3).

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